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Locales and Sheaf Representations of Rings

by

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Mathematics.

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Chapter 0

Introduction

0.1 Historical background

“Abstraction by deletion is a straightforward process: One carefully omits parts of the data describing the mathematical concept in question to obtain the more ‘abstract’ concept. This often leads to a reverse process, in which it is shown that all (or some) of the abstract objects can have the deleted data restored, perhaps in more than one way. Such a restoration is then called a ‘representation theorem’.” - S. Mc Lane, *Mathematics Form and Function*, pp.436

The above paragraph describes, intuitively, the idea of representation theory. Examples of representation abound. In 1878 Cayley showed that every abstract group is abstractly isomorphic to a ‘concrete’ group of permutations; around 1935 logicians A.Lindenbaum and A.Tarski showed that any complete, atomic Boolean algebra is isomorphic to the algebra of all subsets of some set. However, not so much has been done beyond this without restrictions to particular cases - e.g. the algebra concerned should be finite/or atomic, etc.

The breakthrough was then reached with the work of Marshall Stone, see introduction to [11]. Stone’s work were published in two papers in the Transaction of the American Mathematical Society in 1936, 1937. Resulting from these are, among others, *Stone’s representation theorem*, see [11]II,4.4 and *Stone’s [1940] version of the Real Gelfand duality theorem*, see [11] IV,4.

The main tools in Stone’s work were

1. The identification of Boolean algebras with Boolean rings, i.e. rings in which every element, a , satisfies $a^2 = a$, see [11] I,1.8. This led to his

realization of the importance of ideals, especially prime ideals, in lattice theory.

2. The introduction of topology: For a given Boolean algebra A , Stone considered the set

$$\text{Spec}A = \{I \subseteq A \mid I \text{ a prime ideal of } A\}.$$

He then topologized $\text{Spec}A$ by taking the basic opens to be subsets of the form

$$\Omega_a = \{I \in \text{Spec}A \mid a \notin I\}.$$

Then, the opens of $\text{Spec}A$ correspond to arbitrary ideals of A . In this topology the *clopen* sets correspond to principal ideals, and hence to elements of A . Hence a representations of elements of A as continuous maps $\text{Spec}A \rightarrow \mathbf{2}$, where $\mathbf{2}$ is the two element set $\mathbf{2} = \{0, 1\}$ with discrete topology.

However, not so many types of rings can be built up as rings of continuous functions on $\text{Spec}A$ as described above, with values in a single ‘standard’ ring \mathbf{R} . The rescue here is the use of Sheaf theory, where we are instead trying to present a given ring A as a ring of *global sections*, i.e. $A \cong \mathcal{A}(\text{Spec}A)$, for some sheaf of rings \mathcal{A} on $\text{Spec}A$.

On the other hand the spaces $\text{Spec}A$, as described above, will only be useful if one assumes the *Prime Ideal Theorem* that any non-trivial Boolean algebra has a prime ideal, see [7] chapter 9: otherwise $\text{Spec}A$ may well be empty and thus utterly fail to carry any information concerning A . The rescue to this end is the pointfree version, where instead of the topological space $\text{Spec}A$ we consider the *locale*, also call it $\text{Spec}A$, given by the *frame* $\text{Rid}A$ of radical ideals of A . $\text{Rid}A$ constitute the pointfree antecedent of $\text{Spec}A$ regardless of any choice assumption.

0.2 Synopsis

The main purpose of this thesis is to present the (generic) pointfree version of the usual Grothendieck Sheaf representation of a commutative ring with unit.

Chapter one deals with locales/frames. These are ‘generalized’ spaces, where the ‘space’ is not determined by its points, but by the lattice of its opens.

In chapter two we build up some ring theory, needed for the presentation of the sheaf representation of rings. In particular we show that $\text{Rid}A$ is a frame, hence giving us a locale $\text{Spec}A$, for a given ring A .

In chapter three we present the pointfree version of the usual Gelfand sheaf representation of a ring.

In chapter four we again present the pointfree Sheaf representation of a ring as in chapter three, but this time in a generic way.

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Chapter 1

Locale(Frame) Theory

1.1 Introduction and background

In this chapter we recall basic facts about locales(frames). Frames are lattices which possess the essential properties of topological spaces : they are complete, allowing for unions and intersections, and their finite intersections are distributive over arbitrary unions: For any frame L

$$a \wedge \bigvee S = \bigvee \{a \wedge t \mid t \in S\} \quad (a \in L, S \subseteq L).$$

Frame homomorphisms similarly abstract the key properties of inverse image mappings between topologies induced by continuous functions, namely preservation of arbitrary unions, including the zero, and finite intersections, including the unit. This defines the category **Frm**.

In locale theory, we stipulate that the frame view of topology is the complete picture. The result is a model of the notions of space and continuity which is in essence similar to that of point set-topology, but which avoids reducing the “substance” of a space to atoms.

Rather than introducing the category **Loc** of locales as the formal dual of the category **Frm** of frames, we say - in the spirit of categorical language - that there is a functor

$$\mathbf{Loc}^{op} \longrightarrow \mathbf{Frm},$$

denoted by $\Omega^{(-)}$, which is an isomorphism. Thus any frame is of the form Ω^X , where X is the unique locale specified by Ω^X . Similarly, any frame homomorphism $\Omega^X \longrightarrow \Omega^Y$ is of the form Ω^f for a unique morphism $Y \longrightarrow X$ of locales, the map specified by Ω^f . We will denote Ω^X by ΩX (and call it the frame of opens of X) and Ω^f by f^* , and write f_* for the right adjoint of Ω^f . f_* exists since f^* preserves colimits(\bigvee). We will be free to do our calculations in either of the two categories.

Here we would like to recall some basic facts from sheaf theory. We intend to recall only concepts that we will need later in our work and thus will not go deep into the subject.

Definition 1.1.1 A presheaf on a locale X is a contravariant functor from ΩX to *Sets*. A presheaf is thus specified by giving a set $F(U)$ for each $U \in \Omega X$, and a map $\rho_V^U : F(U) \rightarrow F(V)$, for each $V \subseteq U$ in ΩX ; with $\rho_U^U = \text{id}$ for any $U \in \Omega X$, and $\rho_W^U = \rho_W^V \circ \rho_V^U$ for any $W \leq V \leq U$ in ΩX . The maps ρ_V^U are called restriction maps (or simply restrictions). A morphism of presheaves is just a natural transformation of functors.

Notation: We will write $x|V$ for $\rho_V^U(x)$, $x \in F(U)$, $V \subseteq U$.

Definition 1.1.2 Let F be a presheaf on a locale X and let $(U_i)_{i \in I}$ be a family of elements of ΩX . A family $(x_i \in F(U_i))_{i \in I}$ is compatible when

$$\forall i, j \in I \quad x_i|U_i \wedge U_j = x_j|U_i \wedge U_j.$$

Definition 1.1.3 A presheaf F on a locale X is separated when, given $U = \bigvee_I U_i$ in ΩX and $x, y \in F(U)$ in F

$$(\forall i \in I \quad x|U_i = y|U_i) \implies x = y.$$

Definition 1.1.4 A presheaf F on a locale X is a sheaf when, given $U = \bigvee_I U_i$ in ΩX and $(x_i \in F(U_i))_I$ a compatible family in F , there exists a unique $x \in F(U)$ such that for each index $i \in I$, $x|U_i = x_i$.

The condition in definition 4 is called the *patching (glueing) condition*. Note that any sheaf is separated. Diagrammatically, we see that a presheaf F is a sheaf if for any $U = \bigvee_I U_i$ in ΩX , the diagram

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \wedge U_j)$$

is an equaliser, where the maps are induced by restrictions in the obvious way.

Let F be a presheaf on a locale X , and let $U \in \Omega X$. Define a relation \sim on $F(U)$ by $x \sim y$ iff there exists a cover $U = \bigvee_I U_i$ in ΩX with $x|U_i = y|U_i$ for all $i \in I$. It can be easily verified that \sim is an equivalence relation. Then the presheaf $U \mapsto F(U)/\sim$ is separated and gives a reflection of F in the category of separated presheaves on X .

Now, suppose F is a separated presheaf on X . For each $U \in \Omega X$ let $\tilde{F}(U)$ = the set of equivalence classes of compatible families $(x_i \in F(U_i))_I$, where $U = \bigvee_I U_i$ in ΩX , with two families $(x_i \in F(U_i))_I$ and $(y_j \in F(V_j))_J$ equivalent iff $x_i|U_i \wedge V_j = y_j|U_i \wedge V_j$ for all indices i and j . Then $U \mapsto \tilde{F}(U)$ is a sheaf on X , and it is the reflection of F in the category of sheaves on X .

Of particular interest is the following. Let A be a set and X a locale. Consider the presheaf $U \mapsto A$ on X (i.e. the constant presheaf equals to A). It can be easily verified that the sheaf reflection, \hat{A} , of this sheaf is given by $\hat{A}(U) =$ the set of locally constant locale maps $U \rightarrow A$, which are just frame homomorphisms $\mathcal{P}A \rightarrow \downarrow U$ (where $\mathcal{P}A$ stands for the power set of A). Further, for $V \leq U$ in ΩX , the restriction map $\rho_V^U : \hat{A}(U) \rightarrow \hat{A}(V)$ is given by $\alpha \mapsto ((-) \wedge V) \circ \alpha$.

Definition 1.1.5 A locale X is called *discrete* if $\Omega X = \mathcal{P}S$ for some set S . (Where $\mathcal{P}S$ stands for the power set of S).

Definition 1.1.6 A subframe of a frame L is a subset of L which is closed under \wedge and \vee . A quotient of L is an onto (surjective) map $L \longrightarrow M$ in **Frm**.

The fact that the binary meet preserves (distributes over) joins is equivalent to the existence, for each $a \in L$, of a right adjoint $a \rightarrow (-) : L \longrightarrow L$ to the mapping $a \wedge (-) : L \longrightarrow L$ i.e. $a \wedge b \leq c \iff b \leq a \rightarrow c$. Explicitly, $a \rightarrow (-)$ is given by

$$a \rightarrow c = \bigvee \{b \in L \mid a \wedge b \leq c\}.$$

$a \rightarrow c$ is called the *relative pseudocomplement* of a with respect to c . Thus L is a complete Heyting algebra, see [11], chapter 1.

By a semilattice we will mean a meet semilattice with a unit. Semilattice homomorphisms are maps between semilattices which preserve \wedge and the unit. This gives us the category **Slatt** of semilattices and their homomorphisms.

Proposition 1.1.1 Given a set X , the set FX of all finite subsets of X , ordered by \supseteq , is (up to isomorphism) the free semilattice on X , with $\wedge = \cup$ and the universal map $X \xrightarrow{\{-\}} FX, x \mapsto \{x\}$.

Proof Suppose $A, B \subseteq X$ are finite subsets. Then, clearly, $A \cup B$ is finite. Furthermore, $A \cup B$ is the smallest finite subset of X containing both A and B . It is then clear that FX is a semilattice. For any map $X \xrightarrow{h} L$ from X to a semilattice L , define $\bar{h} : FX \longrightarrow L$ by $\bar{h}(A) = \bigwedge h[A]$. Then, $\bar{h}(A \cup B) = \bigwedge h[A \cup B] = \bigwedge (h[A] \cup h[B]) = (\bigwedge h[A]) \wedge (\bigwedge h[B]) = \bar{h}(A) \wedge \bar{h}(B)$. Where the second last equality follows from the fact that A and B are finite sets. Moreover $\bar{h}(\emptyset) = \bigwedge h[\emptyset] = \bigwedge \emptyset = 1$, showing that \bar{h} is a semilattice homomorphism. Further, $\bar{h}(\{x\}) = \bigwedge h[\{x\}] = \bigwedge \{h(x)\} = h(x)$, i.e. the diagram

$$\begin{array}{ccc} X & \xrightarrow{\{-\}} & FX \\ & \searrow h & \downarrow \bar{h} \\ & & L \end{array} \quad \text{commutes. } \square$$

Theorem 1.1.1 For each set X there exists a frame free over X .

Proof As a result of proposition 1, it suffices to give a “free” functor $\mathbf{Slatt} \rightarrow \mathbf{Frm}$. Given any meet semilattice A , let $\mathcal{D}A$ be the set of all downsubsets of A . We show that $\mathcal{D}A$ is the free frame on A , with the universal map $A \rightarrow \mathcal{D}A$, $a \mapsto \downarrow a$. $\mathcal{D}A$ ordered by \subseteq is clearly a sub-complete lattice of $\mathcal{P}A$, and hence also the infinite distributive law holds. Moreover, $\downarrow(a) \cap \downarrow(b) = \downarrow(a \wedge b)$ and $\downarrow(1_A) = A = 1_{\mathcal{D}A}$; so $\downarrow(-)$ is a semilattice homomorphism. Given any meet semilattice homomorphism $A \rightarrow B$, where B is a frame, define $\bar{h} : \mathcal{D}A \rightarrow B$ by $\bar{h}(S) = \bigvee h[S]$. Then

$$\bar{h}(\downarrow a) = \bigvee h[\downarrow a] = \bigvee \{h(x) \mid x \leq a\} = h(a).$$

i.e. the diagram

$$\begin{array}{ccc} A & \xrightarrow{\downarrow(-)} & \mathcal{D}A \\ & \searrow h & \downarrow \bar{h} \\ & & B \end{array}$$

commutes. It is clear from the definition of \bar{h} that it preserves arbitrary joins. For finite meets we have: Let $S, T \in \mathcal{D}A$.

$$\begin{aligned} \bar{h}(S \cap T) &= \bigvee h[S \cap T] \\ &= \bigvee \{h(u) \mid u \in S \cap T\} \\ &= \bigvee \{h(s \wedge t) \mid s \in S, t \in T\} \\ &= \bigvee \{h(s) \wedge h(t) \mid s \in S, t \in T\} \\ &= (\bigvee \{h(s) \mid s \in S\}) \wedge (\bigvee \{h(t) \mid t \in T\}) \\ &= \bar{h}[S] \wedge \bar{h}[T]. \end{aligned}$$

The uniqueness of \bar{h} is clear from the fact that any $S \in \mathcal{D}A$ is a join :

$$S = \bigvee \{\downarrow a \mid a \in S\},$$

and this join must be preserved by \bar{h} . \square

Definition 1.1.7 An element $a \in L$ of a frame L is said to be compact (finite) if for every $S \subseteq L$ with $a \leq \bigvee S$, there exists a finite $F \subseteq S$ with $a \leq \bigvee F$. A frame L is called coherent if

- (i) every element of L is a join of compact elements, and
- (ii) The compact elements form a sublattice of L . More specifically, the unit is compact and the binary meet of compact elements is compact.

A locale X is called coherent if ΩX is coherent.

We would like to recall the following facts (for details see [11] or [8]).

Definition 1.1.8 *In any frame M , x is rather below a , written $x \prec a$, if there exists $y \in L$ such that $x \wedge y = 0$ and $y \vee a = 1$. And, L is said to be regular if*

$$\forall a \in L \quad a = \bigvee \{x \in L \mid x \prec a\}.$$

A locale X is said to be regular if ΩX is.

Lemma 1.1.1 *Any homomorphic image of a regular frame is regular. If each $M_i \subseteq L$ is a regular subframe, then the subframe generated by the M_i (i.e. the smallest subframe of L containing all M_i 's) is regular. \square*

As a consequence of lemma 1, any frame L has a largest regular subframe - simply take the subframe generated by all regular subframes of L . Hence the following proposition.

Proposition 1.1.2 *The category \mathbf{RegFrm} of regular frames is coreflective in \mathbf{Frm} with coreflection $\text{Reg}L \subseteq L$, the largest regular subframe of L . \square*

Definition 1.1.9 *A distributive lattice L is called normal if, whenever we are given $b_1, b_2 \in L$ with $b_1 \vee b_2 = 1$, we can find c_1, c_2 with $c_1 \wedge c_2 = 0$, $c_1 \vee b_2 = 1$ and $b_1 \vee c_2 = 1$.*

For the sake of completeness we would like to mention the adjointness between **Top** and **Frm(Loc)**. The lattice $\mathcal{O}X$ of opens of a topological space X is a frame. Furthermore, for any continuous map $f : Y \rightarrow X$ the map $h : \mathcal{O}X \rightarrow \mathcal{O}Y$ defined by $h(U) = f^{-1}(U)$ is a frame homomorphism. This gives a contravariant functor $\mathcal{O} : \mathbf{Top} \rightarrow \mathbf{Frm}$. Conversely, given a frame L , we define the *Spectrum*, ΣL , of L to be the set of all frame homomorphisms $\xi : L \rightarrow \mathcal{P}1$ (where $\mathcal{P}1$ is the frame corresponding to the terminal locale 1) with the topology given by the sets

$$\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}.$$

For $h : M \rightarrow L$, $\Sigma h = \xi \circ h$. Σ and \mathcal{O} are 'adjoint on the right' with adjoint maps $\eta_L : L \rightarrow \mathcal{O}\Sigma L$, $a \mapsto \Sigma_a$ and $\varepsilon_X : X \rightarrow \Sigma\mathcal{O}X$, $x \mapsto \bar{x}$, where $\bar{x}(U)$ is 1 if $x \in U$ and 0 otherwise. Call L *spatial* if η_L is an isomorphism and X *sober* if ε_X is an isomorphism. This adjoint induces a dual equivalence $\mathbf{SpFrm} \cong \mathbf{Sob}$, between the subcategory of spatial frames and sober spaces respectively.

1.2 Images and Parts of a locale

Closure operators and factorization of adjoints:

The theory of monads [16] becomes very simple when specialised to partially ordered sets (as categories), but it remains eminently useful.

Let P be a partially ordered set and consider a *closure operator* (or monad) $t : P \rightarrow P$ on P , a monotone map satisfying $id \leq t$ and $t^2 \leq t$. Let $i : P^t \hookrightarrow P$ be the inclusion of the sub-partially ordered set of fixpoints (i.e. "algebras" or closed elements) for t :

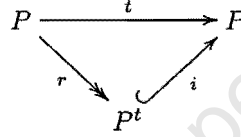
$$x \in P^t \iff t(x) \leq x.$$

Lemma 1.2.1 ($t : P \rightarrow P$ a closure operator on a partially ordered set P)

- (i) t factors as $i \circ r$, where $r : P \rightarrow P^t$ is surjective and left adjoint to i .
- (ii) P^t is closed under all meets which exist in P .
- (iii) r universally co-equalises t and id .

Proof

(i)



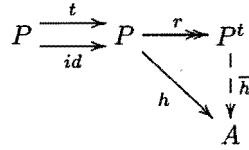
Define r by $r(x) = t(x)$, and let i be the subset inclusion (see the diagram above). Then clearly $i \circ r = t$. For any $x \in P^t$, $x = t(x) = r(x)$, hence r is surjective. Moreover, $r(x) \leq a$ (in P^t) $\iff x \leq r(x) \leq a = i(a)$. Hence $r \dashv i$.

- (ii) Let $S \subseteq P^t$ be such that $\bigwedge S$ exists in P . Then

$$t(\bigwedge S) \leq \bigwedge_{x \in S} t(x) = \bigwedge_{x \in S} x = \bigwedge S.$$

i.e. $\bigwedge S$ is t -closed.

(iii)



$r(t(x)) = t(t(x)) = t^2(x) = t(x) = r(x) = r(id(x))$. Suppose $h : P \rightarrow A$ is a map into a poset A which coequalises t and id . Then, we can define $\bar{h} : P^t \rightarrow A$ by $\bar{h}(z) = h(z)$. \bar{h} is well defined since $h(x) = h(t(x))$ for any $x \in P$. \square

Proposition 1.2.1 *Let*

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} P$$

be an adjunction between partially ordered sets. Then

- (i) $f \circ g \circ f = f$ and $g \circ f \circ g = g$.
- (ii) f is surjective $\iff g$ is injective $\iff f \circ g = id$.
- (iii) $l = g \circ f$ is a closure operator on A , and $A^l = g[P]$.

Proof $f \dashv g \iff [f(a) \leq x \iff a \leq g(x)]$

(i) $f(a) \leq f(a) \iff a \leq gf(a) \iff f(a) \leq fgf(a)$, and $gf(a) \leq gf(a) \iff fgf(a) \leq f(a)$. Thus $f = fgf$. Similarly, from $g \leq g$ and $fg \leq fg$, we get $g = gfg$.

(ii) suppose f is surjective. Let $x, y \in P$, say $x = f(a)$ and $y = f(b)$, with $g(x) = g(y)$. Then $x = f(a) = fgf(a) \leq^* f(b) = y$, where $(*)$ holds since $gf(a) = gf(b)$. Similarly, $y \leq x$. Hence $x = y$, i.e. g is injective. f surjective and $f \circ g \circ f = f \Rightarrow g \circ f = id$.

(iii) Since $f(a) \leq f(a)$ for any $a \in A$, $a \leq gf(a)$. Moreover, $fgf(a) = gf(a)$ from (i). Thus $g \circ f$ is a closure operator. Clearly, $g[P] \subseteq A^l$. Conversely, $a \in A^l \iff a = l(a) = gf(a)$. \square

In a similar way, in proposition 1.2.1, we get that $s = fg$ is a coclosure operator on P and $P^s = f[A]$.

Note that for maps of partially ordered sets $A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} P$, $f \dashv g$ iff $id \leq fg$ and $fg \leq id$.

Proof Only the necessity part needs to be proven. Suppose $id \leq gf$ and $fg \leq id$. Then for any $a \in A$, $x \in P$, $f(a) \leq x \Rightarrow a \leq gf(a) \leq g(x)$ and $a \leq g(x) \Rightarrow f(a) \leq fg(x) \leq x$. \square

Proposition 1.2.2 *Let l be as in the previous proposition and let $t : A \rightarrow A$ be a closure operator. Then the following are equivalent.*

- (i) $t \leq l$
- (ii) g factors through A^t .

proof (i) \implies (ii) Suppose $t \leq l = gf$. Then, for any $a \in A$, a l -closed $\implies a$ is t -closed, i.e. $A^t = g[P] \subseteq A^t$. Thus $i \circ g = g$ where $i : A^t \hookrightarrow A$ is the subset inclusion.

(ii) \implies (i) Suppose g factors through $A^t \hookrightarrow A$. Then $A^t \subseteq A^l$. Then for any $a \in A$ $t(a) \leq tl(a) = l(a)$. \square

For a map of locales $f : Y \rightarrow X$, the operators $f_* f^* : \Omega X \rightarrow \Omega X$ and $f^* f_* : \Omega Y \rightarrow \Omega Y$ preserve finite meets, since both f_* and f^* does.

Definition 1.2.1 A closure operator on a frame L is called a nucleus if it preserves finite meets.

The nucleus f_*f^* , for a locale map $f : Y \rightarrow X$ is called the nucleus induced by f .

Proposition 1.2.3 The following are equivalent for a closure operator $l : L \rightarrow L$ on a frame L .

- (i) $l(a \wedge b) = l(a) \wedge l(b)$
- (ii) $a \rightarrow b \leq l(a) \rightarrow l(b)$
- (iii) $l(a \rightarrow l(b)) = a \rightarrow l(b)$.

Proof The first condition is equivalent to the statement

$$a \wedge b \leq c \Rightarrow l(a) \wedge l(b) \leq l(c),$$

the second to

$$a \wedge b \leq c \Rightarrow a \wedge l(b) \leq l(c)$$

and the third to

$$a \wedge b \leq l(c) \Rightarrow a \wedge l(b) \leq l(c).$$

These statements are clearly equivalent. \square

Let L be a frame, l a nucleus on L , and let

$$L_l = \{a \in L \mid l(a) = a\}.$$

Since $l^2 = l$, $L_l = l[L]$.

Proposition 1.2.4 For a frame L , the following are canonically equivalent.

- (i) The set of quotients of L ,
- (ii) The set of subsets $Q \subseteq L$ closed under arbitrary meets and such that if $a \in L$, $x \in Q$, then $a \rightarrow x \in Q$,
- (iii) The set of nuclei on L .

Proof Let $L \xrightarrow{q} Q$ be a quotient. Then q is surjective and thus $qq_* = id$. Therefore $Q \cong q_*(Q) = L^l$, where l is the nucleus q_*q . If $a \in A$ and $x \in L^l$, then $l(a \rightarrow x) = l(a \rightarrow l(x)) = a \rightarrow l(x) = a \rightarrow x$. We have already seen that L^l is closed under arbitrary meets. Thus $A^l \cong Q$ satisfies (ii). Now suppose $Q \subseteq A$ satisfies (ii). $x \rightarrow y \in Q$ for all $x, y \in Q$ implies that Q is closed under \vee , hence a frame. Q is then a quotient of A with the reflection $L \xrightarrow{q} Q$,

$$a \mapsto \bigwedge \{x \in Q \mid a \leq x\},$$

with the right adjoint the inclusion of Q . Lastly, given a nucleus l on L , $l : L \rightarrow L^l$ gives us a quotient of L . It is not difficult to see that these correspondences are bijections. \square

Definition 1.2.2 A surjection of locales is a map $q : Y \rightarrow Q$ such that $q_*q^* = id$.

Lemma 1.2.2 Given q as in the definition above a map $f : Y \rightarrow X$ factors through q iff $f^*f_* \leq q^*q_*$. The unique factoring map h has $h^* = q_*f^*$. \square

Equivalence classes of surjections $q : Y \rightarrow Q$ ordered by factorization constitute the partially ordered set of *images* of Y .

Note: The images of Y are in bijective correspondence with subframes of ΩY or, equivalently, with meet preserving ‘co-closure operators’ $c : \Omega Y \rightarrow \Omega Y$, both in their usual order.

Definition 1.2.3 A map of locales $e : E \hookrightarrow X$ is called an *embedding* iff $e^* : \Omega X \rightarrow \Omega E$ is a frame surjection, i.e. $e^*e_* = id$.

Lemma 1.2.3 Given e as in the definition above, a map $f : Y \rightarrow X$ factors through e iff $e_*e^* \leq f_*f^*$. The factoring map $g : Y \rightarrow E$ has $g^* = f^*e_*$. \square

The equivalence classes of embeddings $e : E \hookrightarrow X$ ordered by factorization form a set SX , the partially ordered set of *sublocales* of X . It is –partially for psychological comfort– convenient to choose a canonical representative $i : A \hookrightarrow X$ for a sublocale A of X , to which we may then refer as the *sublocale inclusion*, or *part* of X . We do this by taking $i_* : \Omega A \hookrightarrow \Omega X$ to be a subset inclusion. By proposition 4, ΩA is characterised by the equivalent properties of being closed under meets and the operation $u \rightarrow (-)$ for each $u \in \Omega X$, or of being the fixpoints of a meet-preserving closure operator, namely $i_*i^* : \Omega X \rightarrow \Omega X$. Of the correspondences

$$A \leftrightarrow \Omega A \subseteq \Omega X \leftrightarrow \Omega X \xrightarrow{i_*i^*} \Omega X$$

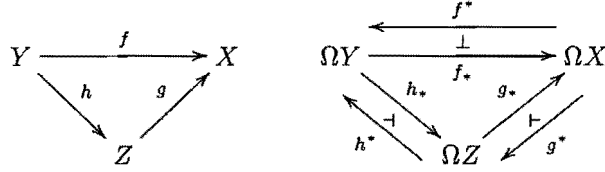
the first is order-preserving and the second order-reversing. The locale X itself is its largest sublocale(part), with the inclusion map the identity $id : X \rightarrow X$.

It can be shown that for that for any locale X , SX is a complete lattice. In fact, it can be shown that SX is a *coframe*(the corresponding locale called a *colocale*), that is

$$\bigwedge \{A \vee S \mid S \in \mathcal{S}\} = A \vee \bigwedge \mathcal{S}$$

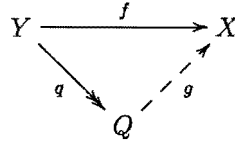
for any $A \in SX$, $\mathcal{S} \subseteq SX$, see [21]. For the purpose of our thesis, we will not go deep into this.

Suppose a locale map $Y \rightarrow X$ factors as



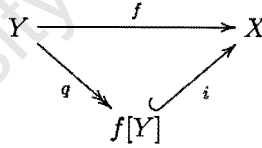
Then for any $u \in \Omega X$, $g_* g^*(u) \leq g_*(h_* h^* g^*(u)) = f_* f^*(u)$. On the other hand, for any $v \in \Omega Y$, $f^* f_*(v) \leq f^* g_* g^*(f_*(v)) = h^* h_*$. Thus $g_* g^* \leq f_* f^*$ and $f^* f_* \leq h^* h_*$.

'Conversely', suppose that in the diagram



q is a surjection ($q_* q^* = id$), and that $f^* f_* \leq q^* q_*$. Then we can define $g : Q \rightarrow X$ by $g^* = q_* f^*$. We then get $q^* g^* = q^* q_* f^* = id \circ f^* = f^*$. Thus f factors through q .

Proposition 1.2.5 *Any map $f : Y \rightarrow X$ of locales factors as a surjection followed by a sublocale inclusion.*



Proof Define $\Omega f[Y] := f_*[\Omega Y]$. It can then be easily seen, using the previous two lemmas that $f^* f_* = q^* q_*$ and $i_* i^* = f_* f^*$. \square

We call $f[Y]$, in the proposition above, the *image of f* .

Remark. $f[Y]$ is the smallest sublocale of X through which f factors. More generally, if $Y_l \hookrightarrow Y$ is a sublocale of Y , given by the nucleus l on ΩY , then its image $f[Y_l]$ is defined to be the sublocale of X given by the nucleus $f_* l f^*$ on ΩX .

It follows purely formally that any of the following is an embedding: any pullback of an embedding, any map with a left inverse (a split mono or section) and, more generally, any equaliser.

We recall that a point $(x : 1 \rightarrow X)$ of a locale X is a frame homomorphism $\Omega X \rightarrow \mathcal{P}1$. Any such point is an embedding since it splits the unique morphism $! : X \rightarrow 1$. We call the image of $x : 1 \rightarrow X$ a *singleton* or *atomic sublocale* of X , denoted $\{x\}$. We label the point of 1 by "0", so that we may write $1 = \{0\}$.

Each $u \in \Omega X$ gives sublocales

$$X_{\downarrow u} \xrightarrow{i} X \quad \text{and} \quad X_{\uparrow u} \xrightarrow{j} X$$

defined by $\Omega X_{\downarrow u} = \downarrow u$ and $\Omega X_{\uparrow u} = \uparrow u$ respectively.

Lemma 1.2.4 *The inclusions $X_{\downarrow u} \xrightarrow{i} X$ and $X_{\uparrow u} \xrightarrow{j} X$ have $i_* i^* = u \rightarrow (-)$ and $j_* j^* = u \vee (-)$ respectively.*

Proof It is clear that $\downarrow u$ and $\uparrow u$ are frames and that the two closure operators in question are nuclei. $w \in \uparrow u \Leftrightarrow w \geq u \Leftrightarrow w = u \vee w$. Thus $\uparrow u$ is (isomorphic) to the frame of closed elements for $u \vee (-)$. For $\downarrow u$ see [11]. \square

Definition 1.2.4 *The Sublocales $X_{\uparrow u} \xrightarrow{j} X$ and $X_{\downarrow u} \xrightarrow{i} X$ are called closed sublocale and open sublocale respectively. Similarly, we call the nuclei $u \vee (-)$ and $u \rightarrow (-)$ closed and open respectively.*

It follows that for a map of locale $f : Y \rightarrow X$ and an open sublocale $Y_{\downarrow v} \xrightarrow{i} Y$ of Y , its image $f[Y_{\downarrow v}]$ is given by the nucleus $f_*(v \rightarrow f^*(-))$ on ΩX .

Notations: For $u \in \Omega X$ we will denote the corresponding open sublocale of X by U (or $U \hookrightarrow X$).

1.3 Direct and inverse images

Let $f : Y \rightarrow X$ be a locale morphism. A part B of Y has a *direct image* $f[B]$ under f in X obtained by factoring the restriction $f|_B$ of f to B into a surjection followed by a sublocale inclusion:

$$\begin{array}{ccc} B & \xrightarrow{\quad} & Y \\ \downarrow & & \downarrow f \\ f[B] & \xrightarrow{\quad} & X. \end{array}$$

The assignment $B \mapsto f[B]$ defines an order preserving mapping $SY \rightarrow SX$. Suppose B is the union $\bigvee B$ of a family \mathcal{B} of parts of Y , $B = \bigvee \{B_i \mid i \in I\}$. The restriction of a surjection to a cover remains a cover, which implies

$$f[B] = \bigvee \{f[B_i] \mid i \in I\}.$$

Where by a the restriction of $f : Y \rightarrow X$ to a family $\mathcal{Y} \equiv \{y_i : Y_i \rightarrow Y \mid i \in I\}$, we mean the family $\{f \circ y_i : Y_i \rightarrow X \mid i \in I\}$. Thus the direct image $f[-] : SY \rightarrow SX$ preserves unions, hence it has a right adjoint. Its right adjoint assigns to $A \in SX$ the largest part $f^{-1}A$ of Y mapped into A by f , namely the inverse image of A along f . A map $g : Z \rightarrow Y$ clearly factors through $f^{-1}A$ if and only if $f \circ g$ factors through A — this says that the diagram

$$\begin{array}{ccc} f^{-1}A & \hookrightarrow & Y \\ \downarrow & & \downarrow f \\ A & \hookrightarrow & X \end{array}$$

is a pullback.

The composition of a map $g : Y \rightarrow A$ with an inclusion $A \hookrightarrow X$ is an inclusion if and only if g is. If $f : Y \rightarrow X$ is a sublocale inclusion, then composition with f turns parts of Y into parts of X - we may view this as an inclusion mapping $SY \hookrightarrow SX$. In the diagram above, $f^{-1}A$ then denotes the intersection $Y \wedge A$ in the lattice SX , with $f^{-1}A \rightarrow A$ and $f^{-1}A \rightarrow Y$ inclusion maps.

1.4 Open maps of locales

Definition 1.4.1 *A map of locales $f : Y \rightarrow X$ is called open if f^* admits a left adjoint $f_!$ and the "Frobenius identity"*

$$f_!(v \wedge f^*(u)) = f_!(v) \wedge u$$

is satisfied for all $u \in \Omega X$, $v \in \Omega Y$.

Proposition 1.4.1 *Let $f : Y \rightarrow X$ be a map of locales. Then the following are equivalent.*

- (i) *f is open;*
- (ii) *f^* preserves arbitrary meets and the identity*

$$f^*(u \rightarrow w) = f^*u \rightarrow f^*w$$

holds for all $u, w \in \Omega X$.

- (iii) *f^* admits a left adjoint $f_!$ and the identity*

$$f_*(v \rightarrow f^*(u)) = f_!(v) \rightarrow u$$

holds for all $u \in \Omega X$, $v \in \Omega Y$.

Proof (i) \Leftrightarrow (ii) f^* has a left adjoint iff f^* preserves arbitrary meets. The Frobenius identity means that for any $u \in \Omega X$ the square

$$\begin{array}{ccc} \Omega Y & \xrightarrow{f_!} & \Omega X \\ (-) \wedge f^* u \downarrow & & \downarrow u \wedge (-) \\ \Omega Y & \xrightarrow{f_!} & \Omega X \end{array}$$

commutes. But this square commutes iff the square of right adjoints

$$\begin{array}{ccc} \Omega Y & \xleftarrow{f^*} & \Omega X \\ f^* u \rightarrow (-) \uparrow & & \uparrow u \rightarrow (-) \\ \Omega Y & \xleftarrow{f^*} & \Omega X \end{array}$$

commutes, which is the identity in (ii).

(i) \Leftrightarrow (iii) The Frobenius identity can also be expressed by saying that for any $v \in \Omega Y$ the square

$$\begin{array}{ccc} \Omega Y & \xleftarrow{f^*} & \Omega X \\ v \wedge (-) \downarrow & & \downarrow f_!(v) \wedge (-) \\ \Omega Y & \xrightarrow{f_!} & \Omega X \end{array}$$

commutes, and again, this is so iff the corresponding square of right adjoints

$$\begin{array}{ccc} \Omega Y & \xrightarrow{f_*} & \Omega X \\ u \rightarrow (-) \uparrow & & \uparrow f_!(v) \rightarrow (-) \\ \Omega Y & \xleftarrow{f^*} & \Omega X \end{array}$$

commutes, which is the identity in (iii). \square

Proposition 1.4.2 *A map of locales $f : Y \rightarrow X$, is open iff for any open sublocale $V \hookrightarrow Y$, its image $f[V] \hookrightarrow X$ is open.*

Proof (\Rightarrow) : Suppose f is open. Recall that $f[V]$ is given by the nucleus $f_*(v \rightarrow f^*(-))$. By proposition 1(iii) $f_*(v \rightarrow f^*(-)) = f_!(v) \rightarrow (-)$. Thus $f[V]$ is open.

(\Leftarrow) : Conversely, suppose $f[V]$ is open for any $v \in \Omega Y$. Then, for any $v \in \Omega Y$ there exists $w \in \Omega X$ such that

$$f_*(v \rightarrow f^*(-)) = w \rightarrow (-).$$

For any $v \in \Omega Y$, let $f_!(v) = w$. Then for any $u, u' \in \Omega X$ $f_!v \wedge u' \leq u \Leftrightarrow u' \leq f_!v \rightarrow u \Leftrightarrow u' \leq f_*(v \rightarrow f^*(u)) \Leftrightarrow f^*u' \leq v \rightarrow f^*u \Leftrightarrow f^*u' \wedge v \leq f^*u$. In particular, taking $u' = 1$, we get $f_!v \leq u \Leftrightarrow v \leq f^*u$. Hence $f_! \dashv f^*$. \square

Corollary 1.4.1 $f : Y \rightarrow X$ is open iff the direct image map $SY \xrightarrow{f[-]} SX$ restricts to open sublocales i.e. the diagram

$$\begin{array}{ccc} \Omega Y & \hookrightarrow & SY \\ \downarrow & & \downarrow f[-] \\ \Omega X & \hookrightarrow & SX \end{array}$$

can be filled in so as to commute.

Corollary 1.4.2 The composite of two open maps is open.

Proof Obvious from proposition 1(ii) or directly from proposition 2. \square

Example A sublocale $i : E \hookrightarrow X$ is open iff i is an open map.

Proof (\Rightarrow .) Suppose $i : E \hookrightarrow X$ is an open sublocale. Then $\Omega E \cong \downarrow u$ for some $u \in \Omega X$. Then, $i^* = u \wedge (-)$ and the inclusion $\downarrow u \hookrightarrow \Omega X$ is left adjoint to $u \wedge (-)$ and clearly it satisfies the Frobenius identity. Hence i is an open map. (\Leftarrow .) The converse follows easily from proposition 2, since $id : E \hookrightarrow E$ is an open sublocale of E , and $i[E] \hookrightarrow X$ and $i : E \hookrightarrow X$ are isomorphic sublocales of X . \square

1.5 Etale maps (or local homeomorphisms) of locales

Definition 1.5.1 A map of locales $f : Y \rightarrow X$ is etale when there exist families $v_i \in \Omega Y$, $u_i \in \Omega X$ ($i \in I$) such that

1. $\bigvee_{i \in I} v_i = 1$ and
2. For every index $i \in I$, f restricts to an isomorphism $\downarrow v_i \rightarrow \downarrow u_i$ between the two open sublocales.

Lemma 1.5.1 Let $f : Y \rightarrow X$ be a morphism of locales and let $v \in \Omega Y$, $u \in \Omega X$ be such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & U \\ \downarrow i & & \downarrow j \\ Y & \xrightarrow{f} & X \end{array}$$

commutes; where i and j are sublocale inclusions. Then $g_*(w) = f_*(v \rightarrow w)$ for $w \leq v$, and $g^*(u') = v \wedge f^*(u')$ for $u' \leq u$.

Proof The last equality follows easily since $g^*(u') = g^*(u \wedge u') = v \wedge f^*(u')$ for any $u' \leq u$. For the first inequality we have: For any $w \leq v$, $u \rightarrow g_*(w) = j_*g_*(w) = f_*i_*(w) = f_*(v \rightarrow w)$. Thus $u \wedge f_*(v \rightarrow w) \leq g_*(w)$. But $g_*(w) \leq u$ and $g_*(w) \leq u \rightarrow g_*(w) = f_*(v \rightarrow w)$. Hence $g_*(w) = u \wedge f_*(v \rightarrow w)$ \square .

Proposition 1.5.1 *Every etale morphism of locales is open.*

Proof Let $f : Y \rightarrow X$ be etale. Let $(v_i), (u_i)$ be as in the definition of an etale map. Let $v \in \Omega Y$. Then $v = \bigvee_I v \wedge v_i$. Since $f[-] : SY \rightarrow SY$ preserves unions, $f[V] = \bigvee_I f[V \wedge V_i]$. Moreover, $V \wedge V_i \leq V_i$ and f etale implies (by the previous lemma) that the image $f[V \wedge V_i]$ is an open sublocale of V_i and hence of X . Therefore $f[V]$ is a join of open sublocales and hence is open. \square

Corollary 1.5.1 *A morphism $f : Y \rightarrow X$ of locales is étale iff*

1. f is open, and
2. there exist elements $(v_i \in \Omega Y)$ with $\bigvee_I v_i = 1$, such that for each $i \in I$ the restricted mapping $f_i : V_i \rightarrow f[V_i]$ is an isomorphism of posets.

Proof (\Rightarrow): Is clear. For (\Leftarrow) take the v_i 's as given and take $u_i = f_i(v_i)$ in the definition of an etale map. \square

The following proposition will become very important later in our work.

Proposition 1.5.2 *The following are equivalent for a map of locales $f : Y \rightarrow X$.*

- (1) f is etale
- (2) f is open and the diagonal $Y \xrightarrow{\Delta} Y \times_X Y$ of the kernel pair of f is open as well.

Proof We will not give a complete proof here, but will recall the technicalities involved and give a sketch for the proof. Recall that for any two locales X and Y their product $X \times_l Y$ is given by the frame generated by the pairs $(u, v) \in \Omega X \times \Omega Y$ with respect to the relations

$$(1) \quad (u_1, v_1) \wedge (u_2, v_2) = (u_1 \wedge u_2, v_1 \wedge v_2),$$

$$(2) \quad \bigvee \{(u, v) \mid u \in S\} = (\bigvee S, v) \text{ for all } S \subseteq \Omega X, v \in \Omega Y; \text{ and}$$

$$(3) \quad \bigvee \{(u, v) \mid v \in T\} = (u, \bigvee T) \text{ for all } u \in \Omega X, T \subseteq \Omega Y.$$

We then write $u \times v$ (some people use $u \otimes v$) for the element of $\Omega(X \times_l Y)$ generated by the pair (u, v) . We then get that $u \times v \simeq \pi_X^{-1} u \wedge \pi_Y^{-1} v$ where π_X, π_Y are the projections from $X \times_l Y$ to X and Y respectively.

For any locale X , the diagonal $X \xrightarrow{\Delta} X \times_l X$ is the coequaliser of the pair

$$\begin{array}{ccc} X \times_l X & \xrightarrow{\pi_1} & X \\ \downarrow \pi_0 & & \\ X & & \end{array}$$

i.e. $\pi_0 \circ \Delta = \pi_1 \circ \Delta = id_X$. Δ is given by

$$\Omega(X \times_l X) \xrightleftharpoons[\Delta_*]{\Delta^*} \Omega X$$

where

$$\Delta^*(u \times v) = \bigvee \{w \in \Omega X \mid w \times w \leq u \times v\}$$

and

$$\Delta_*(w) = \bigvee \{u \times v \mid u \wedge v \leq w\}.$$

Since **Loc** has pullbacks, all the above can be done for any map $f : Y \rightarrow X$, by pulling it back against itself, as in the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\Delta} & Y \times_X Y & \xrightarrow{\pi_1} & Y \\ & & \downarrow \pi_0 & & \downarrow f \\ & & Y & \xrightarrow{f} & X \end{array}$$

Back to our theorem: In proving $(i) \Rightarrow (ii)$ we have, by assumption, $Y = \bigvee_I V_i$, with $f : V_i \rightarrow f[V_i]$ an isomorphism for each $i \in I$. Define $\Delta_I : \Omega Y \rightarrow \Delta(Y \times_X Y)$ by

$$\Delta(v) = \bigvee \{(v \wedge v_i) \times (v \wedge v_i) \mid i \in I\}.$$

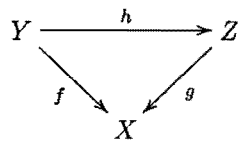
One can then show that $\Delta_I \dashv \Delta^*$ and that the Frobenius identity holds.

In proving $(ii) \Rightarrow (i)$ we consider $\Delta_I(1) \in \Omega(Y \times_X Y)$. One can then show that

$$\Delta_I(1) = \bigvee \{u_i \times w_i \mid i \in I\}$$

for some indexing set I . We then take $v_i = u_i \wedge w_i$ in the definition of an etale map. The reader is referred to [5, 22, 13]. \square

Theorem 1.5.1 *In Loc consider a commutative diagram*



If f and g are étale, h is étale as well. \square

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Chapter 2

Ring Theory

2.1 Some (more) lattice theory

Before we start with rings, let us recall some more lattice concepts. Recall that a complete lattice L is a partially ordered set in which $\bigvee S$ (and hence $\bigwedge S$) exists, for any subset $S \subseteq L$.

A non-empty subset, S , of an ordered set P is said to be *(up)directed* if for every finite $F \subseteq S$, there exists $z \in S$ such that $x \leq z$, for all $x \in F$.

Lemma 2.1.1 *A complete and distributive lattice, L , in which*

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for any $a \in L$, any updirected $S \subseteq L$, is a frame.

Proof For any $X \subseteq L$, let

$$\bigvee X = \bigvee \{x_1 \vee \cdots \vee x_n \mid x_1, \dots, x_n \in X\}.$$

Then the lemma easily follows since the right hand side is the join of an updirected subset of L . \square

Lemma 2.1.2 *Let L be a complete distributive lattice. If $a, b \in L$ are compact elements, then $a \vee b$ is compact. (Thus finite joins of compact elements in L are compact).*

Proof Let a, b be compact. Suppose $a \vee b \leq \bigvee S$ for some $S \subseteq L$. Then $a \leq \bigvee S$ and $b \leq \bigvee S$. Since a and b are compact, there exist finite $T, U \subseteq S$ with $a \leq \bigvee T$ and $b \leq \bigvee U$. Hence $a \vee b \leq \bigvee (T \cup U)$, and $T \cup U$ is finite. \square

2.2 The frame of radical ideals

The reader is supposed to be familiar with the notions of a commutative ring, ideal, prime ideal, maximal ideal, quotient ring, domain, integral domain, field, polynomial ring, ring of fractions [1]; we will recall some other basic notions as we go along. Throughout, unless otherwise stated, by a ring we will mean a commutative ring with a unit. One of the basic notions that play an important role in the pointfree sheaf representation of a ring is that of a radical ideal.

Definition 2.2.1 *An ideal $J \subseteq A$ of a ring A is called radical if $a^n \in J \Rightarrow a \in J$, all $a \in A, n \geq 0$.*

For any ideal $J \subseteq A$, there is the smallest radical ideal, $\rho(J)$, that contains it. As can be easily seen from the definition of a radical ideal, $\rho(J)$ is given by

$$\rho(J) = \{x \in A \mid x^n \in J \text{ for some } n\}.$$

Notation: For any $a \in A$ we will denote $\rho(Aa)$ by $[a]$. Thus

$$[a] = \{x \in A \mid x^n = ac, \text{ some } c, n\}.$$

Note that for any $a \in A$ and $n > 0$, $[a^n] = [a]$.

Denote by $RidA$ the set of all radical ideals of a ring A , partially ordered by inclusion. We show that $RidA$ is a frame.

Let $(J_\alpha)_{\alpha \in \Gamma}$ be a family of radical ideals of A . Let $x^n \in \bigcap J_\alpha$, for some $x \in A$ and some integer n . Then $x^n \in J_\alpha$ for all $\alpha \in \Gamma$. Thus $x \in J_\alpha$ for all $\alpha \in \Gamma$, and hence $x \in \bigcap J_\alpha$. This shows $RidA$ has arbitrary meets given by \bigcap . The bottom is $[0]$.

Let \mathcal{X} be a collection of radical ideals of A . We already know that $\sum \mathcal{X}$, which is the set of all finite sums of elements from members of \mathcal{X} , is the smallest ideal containing all elements of \mathcal{X} . But $\sum \mathcal{X}$ is not necessarily radical. Therefore the join of \mathcal{X} in $RidA$ is

$$\bigvee \mathcal{X} = \rho(\sum \mathcal{X}).$$

The top(unit) is $[1] = A$.

For distributivity, we have the following. Two cases of joins: (i) If we have a directed family of radical ideals, then their join is just their union, and hence the frame distributive law holds since \cap distributes over \bigcup . (ii) For finite joins, reduced by induction to the finitary case:

$$I \wedge (J \vee H) = I \cap \rho(J + H).$$

Therefore,

$$\begin{aligned} a \in I \wedge (J \vee H) &\Rightarrow \text{some } a^n = b + c, \quad b \in J, c \in H \\ &\Rightarrow a^{n+1} = ab + ac \in I \cap J + I \cap H. \end{aligned}$$

Which implies $a \in (I \wedge J) \vee (I \wedge H)$, showing that $I \wedge (J \vee H) \subseteq (I \wedge J) \vee (I \wedge H)$. The reverse inclusion is obvious. Thus $RidA$ is a frame.

For any $a, b \in A$, $[a] \cap [b] = [ab]$: Obviously $[ab] \subseteq [a] \cap [b]$. Conversely, if $x^n = ac$ and $x^m = bd$, then $x^{m+n} = x^m x^n = (ac)(bd) = (ab)(cd)$, hence $x \in [ab]$. Thus $[ab] = [a] \cap [b]$.

In later chapters we will need the following property of $[-]$: For any $a, b \in A$, $[a + b] \subseteq [a] \vee [b]$: Because $a + b \in Aa + Ab \subseteq \rho(Aa + Ab) = [a] \vee [b]$.

We call $J \in RidA$ *finitely generated* if $J = [a_1] \vee \cdots \vee [a_n]$ for some $n > 0$, some $a_i \in J$.

In $RidA$, J compact iff J is finitely generated: (\Rightarrow) This implication is obvious since for any $J \in RidA$, $J = \bigvee \{[a] \mid a \in J\}$.

(\Leftarrow) Suppose J is finitely generated. Since finite joins of compact elements are compact, it will suffice to look at $J = [a]$. So, suppose $[a] = \bigvee \mathcal{X} = \rho(\sum \mathcal{X})$ for some family \mathcal{X} in $RidA$. Then $a^n = x_1 + \cdots + x_n$ for some $x_i \in J_i \in \mathcal{X}$, some $n > 0$. Which implies $a \in J_1 \vee \cdots \vee J_n$, and thus $[a] = J_1 \vee \cdots \vee J_n$.

Since for any $J \in RidA$, $J = \bigvee \{[a] \mid a \in J\}$, $RidA$ is a coherent frame.

Lemma 2.2.1 In $RidA$ $[t] \subseteq [r] \vee [s]$ iff $t^n = rc + sd$ for some $n \geq 0$, $c, d \in A$.

Proof (\Leftarrow) : This implication is clear, since

$$\begin{aligned} t^n = rc + sd \in [r] + [s] &\Rightarrow t \in \rho([r] + [s]) = [r] \vee [s] \\ &\Rightarrow [t] \subseteq [r] \vee [s]. \end{aligned}$$

Where the outside r in the second equation means ‘the radical ideal generated by’.

(\Rightarrow) : Conversely, suppose $[t] \subseteq [r] \vee [s] = \rho([r] + [s])$. Then $t^m \in [r] + [s]$ for some $m \geq 0$. i.e. $t^m = x + y$ for some $x \in [r]$, $y \in [s]$. Then, $x^k = ar$ and $y^l = sb$ for some $k, l \geq 0$, $a, b \in A$. We may assume that $k = l$ (take their maximum). Then $x^k = ra$ and $y^k = sb$. Then, using the binomial expansion, we have

$$\begin{aligned} t^{m2k} &= (x + y)^{2k} = \lambda x^k + \mu y^k \\ &= \lambda ra + \mu sb \\ &= rc + sd. \quad \square \end{aligned}$$

Remark More generally, the above lemma can be stated as: $[t] \subseteq [r] \vee [s]$ iff for any $l, p > 0$, $t^n = r^l c + s^p d$ for some $n \geq 0$, $c, d \in A$. This follows from the fact that $[r] = [r^l]$ for any $l > 0$.

2.3 “Freely” inverting an element of a ring

Given a ring A and an element $s \in A$, we consider the problem of ‘freely’ inverting s . This is a fundamental construction and can be carried out as follows. Let $A[X]$ be the *polynomial ring in indeterminate X with coefficients in A* . A is identified with the subring of $A[X]$ consisting of constant polynomials, i.e. we have a ring homomorphism

$$A \xrightarrow{i} A[X], \quad a \mapsto \text{constant polynomial equal to } a,$$

which is simply an inclusion. In $A[X]$ consider the ideal generated by the polynomial $1 - sX$, i.e. all multiples of $1 - sX$ in $A[X]$. We then form the quotient ring $A[X]/(1 - sX)$. We have a canonical map

$$\nu_s : A \hookrightarrow A[X] \twoheadrightarrow A[X]/(1 - sX).$$

Notation: Sometimes we write $A[s^{-1}]$ for $A[X]/(1 - sX)$. We have the following four facts about $\nu_s : A \rightarrow A[s^{-1}]$.

Lemma 2.3.1 ν_s is the universal ring homomorphism from A mapping s to an invertible element.

Proof Since $sX \equiv 1 \pmod{(1 - sX)}$, $\nu_s(s)$ is invertible, namely $\nu_s(s)^{-1} = \nu_s(X)$. Let $\varphi : A \rightarrow B$ be a ring homomorphism with $\varphi(s)$ invertible, say $\varphi(s)b = 1$ for some (necessarily unique) $b \in B$.

$$\begin{array}{ccccc} \nu_s : A & \hookrightarrow & A[X] & \twoheadrightarrow & A[X]/(1 - sX) \\ & \searrow \varphi & \downarrow \bar{\varphi} & \swarrow \bar{\varphi} & \\ & & B & & \end{array}$$

Extend φ to $\bar{\varphi} : A[X] \rightarrow B$ by $\bar{\varphi}(X) = b$. We have $\bar{\varphi}(1 - sX) = \bar{\varphi}(1) - \bar{\varphi}(s)\bar{\varphi}(X) = 0$. Hence $\bar{\varphi}$ factors through $A[X]/(1 - sX)$ by

$$\tilde{\varphi} : A[X]/(1 - sX) \longrightarrow B$$

$$[P] \longmapsto \tilde{\varphi}(P),$$

where $[P]$ = equivalence class mod $1 - sX$ containing P .

We would like to mention here that any element of $A[s^{-1}]$ is of the form $\nu_s(a)\nu_s(s)^{-n}$ for some $a \in A$, some $n \geq 0$. \square

Lemma 2.3.2 $\text{Ker}(\nu_s) = \{a \in A \mid \text{some } s^n a = 0\}$.

Proof $\nu_s(a) = 0 \Leftrightarrow a \equiv 0 \text{ mod } (1 - sX)$ iff there exists $P \in A[X]$, say $P = p_0 + p_1X + \dots + p_nX^n$, ($p_i \in A, p_n \neq 0$), such that

$$\begin{aligned} a &= P(1 - sX) = (p_0 + p_1X + \dots + p_nX^n)(1 - sX) \\ &= (p_0 + p_1X + \dots + p_nX^n) - (p_0sX + p_1sX^2 + \dots + p_nsX^{n+1}). \end{aligned}$$

Since $a \in A$, we get $a = p_0$, $(p_1 - p_0s) = (p_2 - p_1s) = \dots = (p_n - p_{n-1}s) = p_ns = 0$, from which we get $p_1 = p_0s = as$, $p_2 = p_1s = as^2$, $p_3 = p_2s = as^3, \dots, p_n = p_{n-1}s = as^n$. Multiplying the last equation by s we get $0 = p_ns = as^{n+1}$. \square .

Lemma 2.3.3 For any $r, s \in A$, ν_r factors through ν_s iff $[r] \subseteq [s]$.

Proof (\Leftarrow): Suppose $[r] \subseteq [s]$. Then $r^n = sa$ for some a, n . Then

$$\begin{aligned} \nu_r(s)\nu_r(a)(\nu_r(r))^{-n} &= \nu_r(sa)(\nu_r(r))^{-n} = \nu_r(r^n)(\nu_r(r))^{-n} \\ &= (\nu_r(r))^n(\nu_r(r))^{-n} = 1. \end{aligned}$$

Thus, by universality of ν_s , ν_r factors through ν_s .

(\Rightarrow): Conversely, suppose there exist a ring homomorphism $\varphi : A[s^{-1}] \rightarrow A[r^{-1}]$ such that $\varphi \circ \nu_s = \nu_r$.

$$\begin{array}{ccc} A & \xrightarrow{\nu_s} & A[s^{-1}] \\ & \searrow \nu_r & \downarrow \varphi \\ & & A[r^{-1}] \end{array}$$

Then $\nu_r(s)$ is invertible because $\nu_s(s)$ is invertible. Hence there exist polynomials $P(X)$ and $Q[X]$ such that $sP - 1 = (1 - rX)Q$. Let $P(X) = a_0 + a_1X + \dots + a_nX^n$, $Q(X) = b_0 + b_1X + \dots + b_nX^n$ (no loss of generality here). Then from

$$s(a_0 + a_1X + \dots + a_nX^n) - 1 = (1 - rX)(b_0 + b_1X + \dots + b_nX^n)$$

we obtain the relations

$$sa_0 - 1 = b_0, \quad sa_i = b_i - rb_{i-1} \quad (i = 1, \dots, n), \quad rb_n = 0.$$

From which one successively obtains

$$b_0 = sa_0 - 1,$$

$$b_1 = sa_1 + rb_0 = sa_1 + rsa_0 - r,$$

$$b_2 = sa_2 + rsa_1 + r^2sa_0 - r^2,$$

...

$$b_n = sa_n + rsa_{n-1} + r^2sa_{n-2} + \cdots + r^nsa_0 - r^n,$$

and since $rb_n = 0$ we have

$$r^{n+1} = rsa_n + r^2sa_{n-1} + \cdots + r^{n+1}sa_0 = sc$$

for $c = ra_n + r^2a_{n-1} + \cdots + r^{n+1}a_0$, showing that $r \in [s]$. \square

Lemma 2.3.4 (The pullback lemma:) *If $[r] \vee [s] = [t]$ (in $\text{Rid}A$) then the pushout*

$$\begin{array}{ccc} A[t^{-1}] & \xrightarrow{\nu_{st}} & A[s^{-1}] \\ \nu_{rt} \downarrow & & \downarrow \nu_{rs} \\ A[r^{-1}] & \xrightarrow{\nu_{rsr}} & A[(rs)^{-1}] \end{array}$$

is a pullback.

Proof Before we give a proof let us point out that it is clear that the given square is a pushout since $\varphi(r), \varphi(s)$ invertible iff $\varphi(rs)$ invertible for any ring homomorphism φ .

To prove that the given square is a pullback we will first prove it for the case $t = 1$, and then reduce the general case to this case. So, let $[r] \vee [s] = [1] = A$. Let $\nu_r(a)\nu_r(r)^{-n} \in A[r^{-1}]$, $\nu_s(b)\nu_s(s)^{-m} \in A[s^{-1}]$ with $\nu_{rsr}(\nu_r(a)\nu_r(r)^{-n}) = \nu_{rss}(\nu_s(b)\nu_s(s)^{-m})$. Without loss of generality, we may assume $m = n$. Then

$$\begin{aligned} \nu_{rs}(a)\nu_{rs}(r)^{-n} &= \nu_{rs}(b)\nu_{rs}(s)^{-n} \\ \nu_{rs}(a)\nu_{rs}(r)^{-n} - \nu_{rs}(b)\nu_{rs}(s)^{-n} &= 0 \\ \nu_{rs}(a)\nu_{rs}(s)^n - \nu_{rs}(b)\nu_{rs}(r)^n &= 0 \\ \nu_{rs}(as^n - br^n) &= 0. \end{aligned}$$

Therefore, there exists $l \geq 0$ such that $(rs)^l(as^n - br^n) = 0$. i.e. $r^ls^{n+l}a = r^{n+l}s^lb$. From $[s] \vee [r] = [1]$: $1 = r^{n+l}c + s^{n+l}d$, for some $c, d \in A$. Define

$$x = ar^lc + bs^ld.$$

Then

$$\begin{aligned}
r^{n+l}x &= ar^{n+2l}c + br^{n+l}s^l d \\
&= ar^{n+2l}c + ar^l s^{n+l}d \\
&= ar^l(r^{n+l}c + s^{n+l}d) \\
&= ar^l.
\end{aligned}$$

Thus, $\nu_r(r)^{n+l}\nu_r(x) = \nu_r(r^{n+l}x) = \nu_r(ar^l) = \nu_r(a)\nu_r(r)^l$. From which we get $\nu_r(x) = \nu_r(a)\nu_r(r)^{-n}$. Similarly, we can show that $\nu_s(x) = \nu_s(b)\nu_s(s)^{-n}$. It remains to show that x as defined is unique. Suppose $y \in A$ with $\nu_r(y) = \nu_r(x)$ and $\nu_s(y) = \nu_s(x)$. i.e. $\nu_r(y) = \nu_r(a)\nu_r(r)^{-n}$ and $\nu_s(y) = \nu_s(b)\nu_s(s)^{-n}$. Then $r^k(y-x) = 0$ and $s^{k'}(y-x) = 0$, for some $k, k' \geq 0$. We may assume $k = k'$. Then, from $1 = \lambda r^k + \mu s^k$ for some $\lambda, \mu \in A$ (using lemma 2.2.1), we get

$$\begin{aligned}
y-x &= 1(y-x) = (\lambda r^k + \mu s^k)(y-x) \\
&= \lambda r^k(y-x) + \mu s^k(y-x) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Thus x as defined is unique. Now we would like to reduce the general case to this special case that we have just settled.

Suppose $[t] = [r] \vee [s]$ in $\text{Rid}A$. Then $t^m = cr + ds$ for some $m \geq 0, c, d \in A$. Then, in $\text{Rid}(A[t^{-1}])$ we have $[\nu_t(r)] \vee [\nu_t(s)] \supseteq [\nu_t(t)] = [1]$. Thus $[\nu_t(r)] \vee [\nu_t(s)] = [1]$. Then the result follows since

$$A[a^{-1}] \cong (A[b^{-1}])[\nu_b(a^{-1})],$$

for any $a, b \in A$ with $[a] \subseteq [b]$. \square

Chapter 3

Sheaf Representation of Rings

3.1 Introduction

I would like to remind the reader that we have agreed to use the word *ring* to mean a commutative ring with unit, unless otherwise stated. For the definition of a topos the reader is referred to [12] or [17]. By a ring in a topos \mathcal{E} , we mean an object E of \mathcal{E} equipped with the operations

$$E \times E \begin{array}{c} \xrightarrow{+} \\ \xrightarrow{\quad} \end{array} E, \quad E \xrightarrow{-} E \quad \text{and} \quad 1 \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} E,$$

where 1 is the terminal object of \mathcal{E} , subject to the laws

$$(A1) \quad (\forall x \in E) (\forall y \in E) (\forall z \in E) \quad (x + (y + z)) = ((x + y) + z)$$

$$(A2) \quad (\forall x \in E) \quad (x + 0 = x)$$

$$(A3) \quad (\forall x \in E) \quad (x + (-x) = 0)$$

$$(A4) \quad (\forall x \in E) (\forall y \in E) \quad (x + y = y + x)$$

$$(M1) \quad (\forall x \in E) (\forall y \in E) (\forall z \in E) \quad (x(yz)) = ((xy)z)$$

$$(M2) \quad (\forall x \in E) \quad (x1 = x)$$

$$(M3) \quad (\forall x \in E) (\forall y \in E) \quad (xy = yx),$$

where we write xy for $x \cdot y$.

A ring E is said to be local if it satisfies

$$(\forall x \in E) ((\exists y \in E) (xy = 1) \vee (\exists z \in E) ((1 - x)z = 1)) \quad \text{and} \quad \neg(0 = 1).$$

Classically, in *Sets*, this is the same as :

E is nontrivial and $(\forall x \in E) \quad x \text{ or } (1 - x) \text{ is invertible}$,

which is equivalent to :

E has a unique maximal ideal.

For any locale X , a ring in ShX is the same thing as a sheaf \mathcal{A} on X in the category of rings - so each $\mathcal{A}U$, $U \in \Omega X$, is a ring, and the restriction maps $\mathcal{A}U \rightarrow \mathcal{A}V$, $V \leq U$, are ring homomorphisms.

By a *Sheaf representation* of a ring A , we mean a ring \mathcal{A} in ShX for some locale X such that $\mathcal{A}X \cong A$.

Classically, given a ring A we obtain the *Grothendieck* sheaf representation of A as follows. Let

$$SpecA = \{J \subseteq A \mid J \text{ a prime ideal}\}.$$

We then topologize $SpecA$ by taking the basic opens to be the subsets

$$W_a = \{J \in SpecA \mid a \notin J\},$$

for each $a \in A$. The data

$$W_a \mapsto A[a^{-1}]$$

defines a sheaf on the basic opens of $SpecA$. Thus, see [18], there is a ring \mathcal{A} in $Sh(SpecA)$ with $\mathcal{A}SpecA = \mathcal{A}W_1 \cong A[1^{-1}] \cong A$.

However, for this to work we need the *Prime Ideal Theorem*, see [7], otherwise $SpecA = \emptyset$ is possible in which case it will carry no information about A .

We would like to present here the pointfree version of this representation. This representation is on the *locale* $SpecA$ defined by $\Omega(SpecA) = RidA$, for a given ring A .

3.2 Representation

Theorem 3.2.1 *Every ring has a sheaf representation by a local ring on a coherent locale.*

Proof Let A be a ring. For each $J \in RidA$ let

$$\mathcal{A}J = \{(x_s)_{s \in J} \in \prod_{s \in J} A[s^{-1}] \mid \text{if } [r] \subseteq [s] \text{ then } x_r = \nu_{rs}(x_s)\}$$

and let the restriction maps $\mathcal{A}J \rightarrow \mathcal{A}I$, $I \subseteq J$ in RidA , be projections to partial products.

$\mathcal{A}J$ is a ring, for any $J \in \text{RidA}$:

$\prod_{s \in J} A[s^{-1}]$ is taken as the product ring in the usual sense, and $\mathcal{A}J$ is then easily seen to be a subring of this. Further, the restriction maps defined above are evidently ring homomorphisms which satisfy the required conditions to make \mathcal{A} a presheaf. Hence \mathcal{A} is a presheaf of rings on RidA .

\mathcal{A} is a sheaf.

\mathcal{A} is separated: Let $J = \bigvee_I J_i$ in RidA , for some indexing set I .

case 1: If $(J_i)_{i \in I}$ is directed then $J = \bigcup J_i$. Then for any $x = (x_s)_{s \in J} \in \mathcal{A}J$, $x|J_i = 0 \Leftrightarrow x_s = 0$ for all $s \in J_i$, any $i \in I$. Thus

$$\begin{aligned} x|J_i = 0 & \quad \text{for all } i \in I \\ \Leftrightarrow x_s = 0 & \quad \text{for all } s \in J \\ \Leftrightarrow x = (x_s) = (0) = 0 \end{aligned}$$

Case 2: We reduce (by induction) the case of finitary joins to binary joins. Let $J = G \vee H$ in RidA , and let $x \in \mathcal{A}J$. Suppose $x|G = 0$ and $x|H = 0$. For any $s \in J$, $s^n = a + b$ for some $a \in G$, $b \in H$ implies $[s] = [r] \vee [t]$ where $r = as$, $t = bs$, in particular $r \in G$, $t \in H$. By assumption $\nu_{rs}(x_s) = x_r = 0$ and $\nu_{ts}(x_s) = x_t = 0$. Hence, by the Pullback lemma, $x_s = 0$, i.e $x = (0) = 0$.

\mathcal{A} satisfies the glueing (patching) condition: Let $(x^i \in \mathcal{A}J_i)_{i \in I}$ be a family with $J = \bigvee_I J_i$ in RidA , and for any $i, k \in I$ $x^i|J_i \cap J_k = x^k|J_i \cap J_k$.

case 1: suppose the family $(J_i)_{i \in I}$ is directed. Then, in particular, $J = \bigcup_{i \in I} J_i$. Denote the components of x^i by x_s^i so that $x^i = (x_s^i)_{s \in J_i}$. Let $x = (x_s)_{s \in \bigcup_{i \in I} J_i = J}$. x is well defined since the x^i 's agree on the intersections $J_i \cap J_k$, for any $i, k \in I$. Also, it is clear that $x \in \mathcal{A}J$. It is then obvious that x is the glue of the x^i 's.

case 2: Suppose $J = G \vee H$ in RidA . Let $x \in \mathcal{A}G$, $y \in \mathcal{A}H$, ($x = (x_s)_{s \in G}$ and $y = (y_t)_{t \in H}$), with $x|G \cap H = y|G \cap H$. $s \in J \Rightarrow [s] = [r] \vee [t]$ for some $r \in G$, $t \in H$. Since $x|G \cap H = y|G \cap H$,

$$\nu_{rt}(x_r) = x_{rt} = y_{rt} = \nu_{rt}(y_t).$$

Thus, by the pullback lemma, there exists $z_s \in A[s^{-1}]$ such that $x_r = \nu_{rs}(z_s)$ and $y_t = \nu_{ts}(z_s)$.

Given also $[s] = [r'] \vee [t']$ with $r' \in G$, $t' \in H$ and $z'_s \in A[s^{-1}]$ with $x_{r'} = \nu_{r's}(z'_s)$ and $y_{t'} = \nu_{t's}(z'_s)$. We have

$$[s] = [rr'] \vee [rt'] \vee [tr'] \vee [tt'].$$

$z_s, z'_s \in A[s^{-1}]$ and we have

$$\nu_{rr's}(z_s) = \nu_{rr'r}(\nu_{rs}(z_s)) = \nu_{rr'r}(x_r) = x_{rr'}.$$

Similarly we get

$$\begin{aligned}\nu_{rt's}(z_s) &= x_{rt'} \\ \nu_{tr's}(z_s) &= y_{tr'} \\ \nu_{tt's}(z_s) &= y_{tt'}.\end{aligned}$$

Also

$$\nu_{rt's}(z'_s) = \nu_{rt'r}(\nu_{rs}(z'_s)) = \nu_{rt'r}(x_r) = x_{rt'}.$$

Similarly,

$$\begin{aligned}\nu_{rt's}(z'_s) &= y_{rt'} \\ \nu_{tr's}(z'_s) &= x_{tr'} \\ \nu_{tt's}(z'_s) &= y_{tt'}.\end{aligned}$$

Thus, by the pullback lemma, $z_s = z'_s$. Therefore z_s as defined is unique. Let $z = (z_s)_{s \in J} \in \mathcal{A}J$. $z|G = (z_s)_{s \in G}$, but $s \in G \Rightarrow [s] = [s] \vee [0]$ and hence $z_s = x_s$. Thus $z|G = x$ and similarly $z|H = y$. Thus \mathcal{A} is a sheaf of rings on $\text{Rid}A$.

For any $a \in A$, $\mathcal{A}[a] \cong \mathcal{A}[a^{-1}]$:

Define $\varphi : \mathcal{A}[a] \longrightarrow \mathcal{A}[a^{-1}]$ to be the projection

$$x = (x_s)_{s \in [a]} \mapsto x_a.$$

This map is obviously a ring homomorphism. Suppose $\varphi(x) = 0$, i.e. $x_a = 0$. Then for any $s \in [a]$, $x_s = \nu_{sa}(x_a) = \nu_{sa}(0) = 0$, i.e. $x = 0$. Hence φ is one to one. Moreover, for any $b \in \mathcal{A}[a^{-1}]$, define $x \in \mathcal{A}[a]$ by $x_s = \nu_{sa}(b)$. Then $\varphi(x) = x_a = \nu_{aa}(b) = b$, since $\nu_{aa} = \text{id}$. Furthermore, for any $[r] \subseteq [s]$ with $r, s \in [a]$

$$\nu_{rs}(x_s) = \nu_{rs}(\nu_{sa}(b)) = \nu_{ra}(b) = x_r.$$

Thus φ is a bijection. Consequently, we have a sheaf representation of A since

$$\mathcal{A}[1] \cong \mathcal{A}[1^{-1}] \cong A.$$

\mathcal{A} is a local ring :

\mathcal{A} satisfies $\neg(0 = 1)$ since $\mathcal{A}J$ is trivial iff $J = [0]$. Let $x \in \mathcal{A}J$, $J \in \text{Rid}A$. Consider any $s \in J$. Then there exist $a \in A$ and $n \geq 0$ such that for each $t \in [s]$

$$\begin{aligned}x_t &= \nu_{ts}(x_s) = \nu_{ts}(\nu_s(a)\nu_s(s)^{-n}) \\ &= \nu_t(a)\nu_t(s)^{-n},\end{aligned}$$

and hence $x|t = \nu_t(a)\nu_t(s)^{-n}$. Now, for $t = as$, $\nu_t(a)$ is invertible and hence $\nu_t(x) = \nu_t(a)\nu_t(s)^{-n}$ is invertible. For $t = s(s^n - a)$ $\nu_t(s^n - a)$ and hence $\nu_t(1 - x)$ is invertible. Thus we have that $x|[as]$ and $1 - x|[s(s^n - a)]$ are invertible. Further $[s] = [as] \vee [s(s^n - a)]$ and, since $J = \bigvee \{[s] \mid s \in J\}$, this proves

the claim, i.e. $J = G \vee H$ with $x|G$ invertible and $1 - x|H$ invertible. \square

Thus we have obtained the basic result: *Every ring A has a sheaf representation by a local ring on a coherent locale $\text{Spec}A$.*

From now on it will be convenient to denote frames by $\mathcal{L}, \mathcal{M}, \dots$, their elements by U, V, \dots , the zero by 0 and the unit by E . We need the following two lemmas for our next result.

Lemma 3.2.1 *Let \mathcal{L}, \mathcal{M} be frames, and let*

$$\mathcal{L} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{k} \end{array} \mathcal{M}$$

be a pair of frame homomorphisms with $kh = \text{id}_{\mathcal{L}}$ and $hk \leq \text{id}_{\mathcal{M}}$. For any local ring A (in Sheaves) on \mathcal{M} , Ah is a local ring on \mathcal{L} .

Proof Let $a \in AhU$. A local implies $h(U) = G \vee H$ in \mathcal{M} such that $a|G$ and $1 - a|H$ are invertible. Then

$$\begin{aligned} U &= kh(U) = k(G) \vee k(H) \text{ in } \mathcal{L}, \text{ and} \\ hk(G) &\leq G \text{ and } hk(H) \leq H \text{ in } \mathcal{M}. \end{aligned}$$

Thus $a|k(G)$ and $1 - a|k(H)$ are invertible (with respect to the sheaf Ah). \square

Lemma 3.2.2 *Let \mathcal{M} be a compact normal frame and let $\mathcal{L} = \text{Reg}\mathcal{M}$ (see chapter 1). There is a homomorphism $k : \mathcal{M} \rightarrow \mathcal{L}$, given by*

$$k(U) = \bigvee \{V \in \mathcal{M} \mid V \prec U\}.$$

Moreover, $k(U) = U$ for $U \in \mathcal{L}$, i.e. k is a retraction.

Proof It follows easily from the definition of \prec that k preserves 0 , \wedge and E . Further, $k(\bigvee S) = \bigvee k[S]$ for updirected S follows immediately by compactness. Let $V \prec U \vee W$. Then $V^* \vee U \vee W = E$. Therefore, by normality, there exist G, H with $G \wedge H = 0$ such that $U \vee G = W \vee V^* \vee H = E$, and there exist Y, Z with $Y \wedge Z = 0$ such that

$$W \vee Y = E = V^* \vee H \vee Z.$$

As a result, $H \prec U \vee Z \prec W$ and :

$$V = V \wedge (V^* \vee H \vee Z) = (V \wedge H) \vee (V \wedge Z) \leq k(U) \vee k(W).$$

This proves the nontrivial part of $k(U \vee W) = k(U) \vee k(W)$. Thus k is a frame homomorphism. Further, $k(U) \leq U$ is obvious, since $V \prec U \Rightarrow V \leq U$. \square

Definition 3.2.1 A ring A is called Gelfand if $RidA$ is normal.

Remark: The above definition is equivalent to the condition

$$\forall a, b \in A \quad a + b = 1 \Rightarrow (1 + ar)(1 + bs) = 0 \text{ for some } r, s \in A,$$

(see[2]).

Proposition 3.2.1 Every Gelfand ring has a sheaf representation by a local ring on a compact regular locale.

Proof Let A be a Gelfand ring. Since $RidA$ is compact (being coherent) $Reg(RidA)$ is compact regular. Further, since $RidA$ is also normal (by definition) we have a pair of homomorphisms

$$RidA \begin{array}{c} \xrightarrow{k} \\ \xleftarrow{h} \end{array} Reg(RidA),$$

where k is as in lemma 2 and h the identical embedding. Now let \mathcal{A} be the previous representation of A . Then $\mathcal{A}|Reg(RidA)$ is a ring on a compact regular locale, $Reg(RidA)$. By lemma 1, $\mathcal{A}|Reg(RidA)$ is local. \square

I would like to recall here that a ring A is called regular if

$$(\forall x \in A)((\exists y \in A)(x^2y = x))$$

and that a field is a ring satisfying:

$$\neg(0 = 1) \text{ and } (\forall x)((x = 0) \vee (\exists y)(xy = 1)).$$

Theorem 3.2.2 A ring A is regular iff it has a sheaf a sheaf representation by a field on a compact 0-dimensional locale.

Proof Suppose A is regular. Then for any $a \in A$, there exists $b \in B$ such that $a = a^2b$. Then $(ab)^2 = a^2b^2 = a^2bb = ab$, i.e. ab is idempotent. $a = a^2b$ implies $a \in [ab]$. Since $(ab)^2 = ab$, $ab \in [a]$. Thus $[a] = [ab]$. Further, for any idempotent $u \in A$, $[u]$ is complemented in $RidA$: $[u] \wedge [1 - u] = [u(1 - u)] = [0]$ and $[u] \vee [1 - u] = [1]$, the latter since $1 = u + (1 - u)$. Thus $[a]$ is complemented for any $a \in A$. It remains to show that \mathcal{A} is a field. Let $x \in \mathcal{A}J$ $s \in J$ for some $J \in RidA$. Then, there exists $a \in A$ and $n > 0$ such that for each $t \in [s]$, $x_t = \nu_t(a)\nu_t(s)^{-n}$, i.e. $x|s = (\nu_t(a)\nu_t(s)^{-n})$. Now, if $a = a^2b$ and we take $t = abs$, then $\nu_t(a) = \nu_{abs}(a)$ is invertible, and thus $x|[abs] = \nu_{abs}(a)\nu_{abs}(abs)^{-n}$ is invertible, and $x|[s(1 - ab)] = 0$, since $1 - ab$ becomes invertible and $a(1 - ab) = 0$. Conversely, suppose $A \cong \mathcal{A}[1]$ for some field \mathcal{A} on a compact 0-dimensional locale X . Let $s \in \mathcal{A}[1]$. Then, $X = U \vee V$ with $U \wedge V = 0$ such that $s|U = 0$ and $s|V$ invertible, say $s|V \cdot t = 1$ for some $t \in \mathcal{A}V$. Then, $s|U = s^2|U \cdot y$ for any $y \in \mathcal{A}U$ and, on V , $s = s \cdot 1 = sst = s^2t$. Hence A is regular. \square

Chapter 4

The Generic Sheaf Representation of Rings

In this chapter we present the sheaf representation of a ring in a ‘generic’ way. We conclude the chapter with a discussion about different ways of obtaining the frame $\text{Rid}A$ for a given ring A (in Sets); we also mention other ways through which the generic sheaf representation can be obtained.

4.1 Localization

Definition 4.1.1 *Let A be a ring in a topos \mathcal{E} . $P \subseteq A$ is called a prime if*

- (i) $\neg(0 \in P)$.
- (ii) $1 \in P$
- (iii) $ab \in P \Rightarrow a \in P \wedge b \in P$
- (iv) $a + b \in P \Rightarrow a \in P \vee b \in P$.

Thus $P \subseteq A$ is a prime iff the ‘classifying’ map σ in the following pullback is a *support*.

$$\begin{array}{ccc} P & \longrightarrow & 1 \\ \downarrow & & \downarrow t \\ A & \xrightarrow{\sigma} & \Omega \end{array}$$

where Ω is the subobject classifier (truth-value object) of \mathcal{E} ; and by a support we mean that σ satisfies the following properties:

$$\begin{aligned}
\sigma(0) &= 0, \\
\sigma(1) &= 1, \\
\sigma(ab) &= \sigma(a) \wedge \sigma(b) \text{ and} \\
\sigma(a+b) &\leq \sigma(a) \vee \sigma(b).
\end{aligned}$$

Just like in usual commutative algebra, by localization of A at P , we mean constructing a ring homomorphism from A which is universal with respect to mapping P to the (object of) invertible elements; i.e. the universal ring homomorphism $\varphi : A \rightarrow B$ with $\varphi[P] \subseteq \text{Inv}(B) = \{x \in B \mid (\exists y \in B)(xy = 1)\}$.

Proposition 4.1.1 *For any prime P of A the localization at P exists, and is a local ring.*

Proof Define a relation \sim on $A \times P$ by

$$(x, s) \sim (x', s') \text{ iff } (\exists t \in P)(txs' = tx's).$$

\sim is an equivalence relation:

(E1) Since $1 \in P$ $(a, s) \sim (a, s)$ for any $(a, s) \in A \times P$

(E2) Symmetry is clear.

(E3) Suppose $(a, s) \sim (a', s')$ and $(a', s') \sim (a'', s'')$. Then

$$(\exists t_1, t_2 \in P) \quad (t_1as' = t_1a's) \quad (t_2a's'' = t_2a''s').$$

Then $t_1as't_2s'' = t_1a'st_2s'' = t_1st_2a''s'$ and $t_1t_2s' \in P$.

Hence \sim is an equivalence relation.

Notation: We will write $\frac{a}{s}$ for the equivalence class containing (a, s) . We then form a quotient object of $A \times P$ with respect to \sim and denote it by $A[P^{-1}]$. Next we make $A[P^{-1}]$ into a ring by defining addition and multiplication as follows.

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'} \quad \frac{a}{s} \frac{a'}{s'} = \frac{aa'}{ss'}.$$

These operations are well defined as can be easily checked. Also it is easy to check that with these operations $A[P^{-1}]$ is a commutative ring, with the unit $\frac{1}{1}$; the zero is $\frac{0}{1}$ and $-\frac{a}{s} = \frac{-a}{s}$. The universal map $A \rightarrow A[P^{-1}]$ is given by $a \mapsto \frac{a}{1}$.

Moreover, let $(a, s) \in A \times P$. Since $s \in P$, $a + (s - a) \in P$ and therefore $a \in P$ or $s - a \in P$. Then $\frac{s}{a} \in A[P^{-1}]$ or $\frac{s}{s-a} \in A[P^{-1}]$ and $\frac{a}{s} \frac{s}{a} = 1$, $(1 - \frac{a}{s}) \frac{s}{s-a} = 1$. Hence $A[P^{-1}]$ is a local ring. Further, given $\varphi : A \rightarrow B$ such that $\varphi[P] \subseteq \text{Inv}(B)$, define $A \times P \rightarrow B$ by $(x, s) \mapsto \varphi(x)\varphi(s)^{-1}$. It is then easily seen that if $(a, s) \sim (a', s')$ then $\varphi(a)\varphi(s)^{-1} = \varphi(a')\varphi(s')^{-1}$. Thus φ factors through $A[P^{-1}]$. \square

As an example of the above situation, let A be a ring in \mathbf{Sets} , \mathcal{L} a frame and $A \xrightarrow{\sigma} \mathcal{L}$ a support. We would like to show that σ give a prime \hat{P} of \hat{A} (= the sheaf reflection of the constant presheaf \hat{A} , $U \mapsto A$) in $\mathbf{Sh}\mathcal{L}$. We can then localise \hat{A} at \hat{P} to get a local ring $\hat{A}[\hat{P}^{-1}]$.

Let us recall (chapter 1) that $\hat{A}U$ = set of all locally constant locale maps $U \rightarrow A$, which are just the frame homomorphisms $\alpha : \mathcal{P}A \rightarrow \downarrow U$. But these are characterized by their effect on the singletons $\{a\}$, for $a \in A$, since for any $S \in \mathcal{P}A$ $S = \bigvee \{\{a\} \mid a \in S\}$. It then becomes clear that these α 's can be described as maps $\alpha : A \rightarrow \downarrow U$ satisfying

- (i) $\alpha(a) \wedge \alpha(b) = 0$, $a \neq b$ and
- (ii) $\bigvee \{\alpha(a) \mid a \in A\} = U$.

Since A is a ring, we get ring structures on $\hat{A}U$, for each $U \in \mathcal{L}$ as follows

$$(\alpha \square \beta)(a) = \bigvee \{\alpha(b) \wedge \beta(c) \mid b \square c = a\},$$

where \square stands for addition and multiplication. The zero in $\hat{A}U$ is given by

$$\mathbf{0}(a) = \begin{cases} U & : a = 0 \\ 0 & : a \neq 0 \end{cases}$$

$\mathbf{1}$ is given by

$$\mathbf{1}(a) = \begin{cases} U & : a = 1 \\ 0 & : a \neq 1 \end{cases}$$

$-\alpha$ is given by $-\alpha(a) = \alpha(-a)$, for any $\alpha \in \hat{A}U$. These data fits together so as to give a ring structure on $\hat{A}U$ for any $U \in \mathcal{L}$. Furthermore, for any $V \leq U$ in \mathcal{L} , and $\alpha, \beta \in \hat{A}U$,

$$\begin{aligned} ((\alpha \square \beta)|V)(a) &= \bigvee \{\alpha(b) \wedge \beta(c) \wedge V \mid b \square c = a\} = \bigvee \{(\alpha|V)(b) \wedge (\beta|V)(c) \mid b \square c = a\} \\ &= ((\alpha|V) \square (\beta|V))(a), \end{aligned}$$

and similarly for $-\alpha$, $\mathbf{0}$ and $\mathbf{1}$, showing that the restriction maps $\hat{A}U \rightarrow \hat{A}V$ are ring homomorphisms. Thus we have a ring \hat{A} in $\mathbf{Sh}\mathcal{L}$. Note that any $a \in A$ gives rise to $\tilde{a} \in \hat{A}U$, for any $U \in \mathcal{L}$, defined by

$$\tilde{a}(c) = \begin{cases} U & : c = a \\ 0 & : c \neq a \end{cases}$$

Recall that in $\mathbf{Sh}\mathcal{L}$ the object of truth values (=subobject classifier) Ω is given by

$$\Omega U = \downarrow U, \quad U \in \mathcal{L}.$$

The given support $\sigma : A \rightarrow \mathcal{L}$ gives a map $\tilde{\sigma} : \tilde{A} \rightarrow \Omega$, defined at each $U \in \mathcal{L}$ by

$$\tilde{\sigma}_U : \tilde{A}U \rightarrow \Omega U \quad a \mapsto \sigma(a) \wedge U.$$

It can be easily shown that $\tilde{\sigma}$ is a map of presheaves. The ‘unit’ $\eta : \tilde{A} \rightarrow \hat{A}$ is given by

$$\eta_U : \tilde{A}U \rightarrow \hat{A}U, \quad a \mapsto \tilde{a}.$$

at each $U \in \mathcal{L}$. It then follows that the unique map $\hat{\sigma}$ in the following commutative diagram is a support.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\eta} & \hat{A} \\ & \searrow \tilde{\sigma} & \downarrow \hat{\sigma} \\ & & \Omega. \end{array}$$

Explicitly, $\hat{\sigma}$ is given by

$$\hat{\sigma}_U(\alpha) = \bigvee \{ \sigma(a) \wedge \alpha(a) \mid a \in A \}$$

Moreover σ a support implies that $\hat{\sigma}$ is also a support. The point here is that we are now in a topos $Sh\mathcal{L}$ with the object of truth values Ω , and an Ω -valued support $\hat{\sigma} : \hat{A} \rightarrow \Omega$, hence a prime \hat{P} of \hat{A} . Explicitly \hat{P} is given by

$$\hat{P}U = \{ \alpha \in \hat{A}U \mid \hat{\sigma}_U(\alpha) = U \}.$$

As before, we localise \hat{A} at \hat{P} to get a local ring $\hat{A}[\hat{P}^{-1}]$ in $Sh\mathcal{L}$.

Generally, in $Sh\mathcal{L}$, this localization can be briefly described as follows. At each $U \in \mathcal{L}$, we define an equivalence relation \sim on $\hat{A}U \times \hat{P}U$ by $(a, s) \sim (a', s')$ iff there is a cover $U = \bigvee_I U_i$ and a family $(t_i \in \hat{P}U_i)_{i \in I}$ such that $t_i(a|U_i)(s'|U_i) = t_i(a'|U_i)(s|U_i)$. Denote $(\hat{A}U \times \hat{P}U) / \sim_U$ by $Q(U)$. Then $U \mapsto Q(U)$ is a separated presheaf. Indeed, consider $U = \bigvee_I U_i$ and $[a, s], [a', s'] \in Q(U)$ such that $[a, s]|U_i = [a', s']|U_i$ for all $i \in I$. This means $(a|U_i, s|U_i) \sim (a'|U_i, s'|U_i)$ for all $i \in I$. Thus $(a, s) \sim_U (a', s')$ since \sim is a sheaf.

$\hat{A}[\hat{P}^{-1}]$ is the sheaf reflection of Q . By general principles we may consider $Q(U)$ as a subset of $\hat{A}[\hat{P}^{-1}]U$ and note that $\hat{A}[\hat{P}^{-1}]$ is the sheaf generated by this subpresheaf. The image of $(a, s) \in \hat{A}U \times \hat{P}U$ in $\hat{A}[\hat{P}^{-1}]$ will be denoted by $[\frac{a}{s}]_U$.

4.2 The universal support

Classically, i.e. in the category of sets, the primes of a ring A are exactly the complements of prime ideals of A . Hence without the Prime Ideal Theorem (i.e. any nontrivial Boolean algebra has a prime ideal), A need not have any primes. But with the ‘change of toposes’ we can get a generic prime. In chapter 2 we saw that for any ring A in $Sets$, the map $[-] : A \rightarrow RidA$ has the properties

$$\begin{aligned}
[0] &= 0 \\
[1] &= 1 \\
[ab] &= [a] \wedge [b] \text{ and} \\
[a + b] &\leq [a] \vee [b].
\end{aligned}$$

Thus $[-]$ is a support. First we show that $[-]$ is universal. Let $\tau : A \rightarrow \mathcal{L}$ be a support, for some frame \mathcal{L} . Define $\bar{\tau} : \text{Rid}A \rightarrow \mathcal{L}$ by

$$\bar{\tau}(J) = \bigvee \tau[J].$$

Then $\bar{\tau}([a]) = \tau(a)$, for any $a \in A$. We show that τ is a frame homomorphism. $\bar{\tau}(J \wedge I) = \bigvee \tau[J \wedge I] = \bigvee \tau(J \cap I) = \bigvee (\tau(J) \cap \tau(I)) = (\bigvee \tau(J)) \wedge (\bigvee \tau(I))$. Let \mathcal{X} be a directed family of radical ideals of A .

$$\begin{aligned}
\bar{\tau}\left(\bigvee \mathcal{X}\right) &= \bar{\tau}\left(\bigcup_{J \in \mathcal{X}} J\right) = \bigvee (\tau[\bigcup_{J \in \mathcal{X}} J]) = \bigvee (\bigcup_{J \in \mathcal{X}} \tau[J]) \\
&= \bigvee_{J \in \mathcal{X}} (\bigvee \tau[J]) = \bigvee_{J \in \mathcal{X}} \bar{\tau}(J).
\end{aligned}$$

Thus $\bar{\tau}$ is a frame homomorphism.

As shown in the first section above, $[-]$ gives a prime \hat{P} of \hat{A} in $Sh(\text{Rid}A)$. We can then localise \hat{A} at \hat{P} to get a local ring $\hat{A}[\hat{P}^{-1}]$.

4.3 Representation

Let A, \hat{A} , etc. be as in section two above. The aim here is to show that $\hat{A}[\hat{P}^{-1}]$ provides a sheaf representation of A , specifically

$$\hat{A}[\hat{P}^{-1}][a] \cong A[a^{-1}],$$

for any $a \in A$. For any presheaf \mathcal{P} on a frame \mathcal{L} we will denote by $\tilde{\mathcal{P}}$ the sheaf reflection of \mathcal{P} , with the universal map $\eta : \mathcal{P} \rightarrow \tilde{\mathcal{P}}$ from \mathcal{P} to a sheaf; and for any map of presheaves $\tau : \mathcal{P} \rightarrow \mathcal{F}$ where \mathcal{F} is a sheaf, we denote by $\bar{\tau}$ the sheaf extension of τ , i.e. $\bar{\tau}$ is the unique map such that $\bar{\tau} \circ \eta = \tau$. We will need the following lemmas.

Lemma 4.3.1 *Let \mathcal{P} be a presheaf on a frame \mathcal{L} and let $\tau : \mathcal{P} \rightarrow \mathcal{F}$ be a map into a sheaf \mathcal{F} on \mathcal{L} such that the image of τ generates \mathcal{F} and $\text{Im}(\tau)$ is the separated presheaf reflection of \mathcal{P} with the reflection map induced by τ . Then, $\bar{\tau} : \tilde{\mathcal{P}} \rightarrow \mathcal{F}$ is an isomorphism.*

proof

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\eta} & \tilde{\mathcal{P}} \\
& \searrow \tau & \downarrow \bar{\tau} \\
& & \mathcal{F}
\end{array}$$

Since $\tilde{\mathcal{P}}$ is separated and $Im(\tau)$ generates \mathcal{F} , the sheaf reflection map $\eta : \mathcal{P} \rightarrow \tilde{\mathcal{P}}$ factors through τ as $\eta = \xi \circ \tau$. It is then easily seen that $\tilde{\tau} \circ \xi$ and $\xi \circ \tilde{\tau}$ are identities, since η and τ are presheaf epimorphisms. \square .

Lemma 4.3.2 *Let \mathcal{L} be a frame with a basis \mathcal{K} and let \mathcal{P} be a presheaf on \mathcal{L} such that \mathcal{P} is separated and patching at each $W \in \mathcal{K}$. Then $\tilde{\mathcal{P}}W = \mathcal{P}W$ for each $W \in \mathcal{K}$.*

Proof Note that $\tilde{\mathcal{P}}U =$ the set of equivalence classes of compatible families $(x_i \in \mathcal{P}U_i)_I$, where $U = \bigvee_I U_i$ is a cover of U , with two families $(x_i \in \mathcal{P}U_i)_I$ and $(y_j \in \mathcal{P}V_j)_J$ equivalent iff $x_i|U_i \wedge V_j = y_j|U_i \wedge V_j$ for all indices i and j . But, since \mathcal{P} is patching on \mathcal{K} , for each $W \in \mathcal{K}$ and any cover $W = \bigvee_{i \in I} U_i$ of W with compatible $(x_i \in \mathcal{P}U_i)_I$, we have $x_i = x|U_i$ for a unique $x \in \mathcal{P}W$. Conversely, for each $x \in \mathcal{P}W$, $(x|U_i)_I$ is a compatible family, i.e. an element of $\tilde{\mathcal{P}}W$ for each cover $W = \bigvee_I U_i$. \square

Lemma 4.3.3 *Let \mathcal{G}, \mathcal{F} be sheaves on a frame \mathcal{L} and suppose \mathcal{L} has a basis \mathcal{K} such that $\mathcal{F} = \mathcal{G}$ on \mathcal{K} . Then there exists an isomorphism $\mathcal{F} \rightarrow \mathcal{G}$.*

Proof At each $U \in \mathcal{L}$ consider $\mathcal{M}U \subseteq \mathcal{F}U \times \mathcal{G}U$ defined by $(x, y) \in \mathcal{M}U$ iff $x|W = y|W$ for all $W \leq U$ in \mathcal{K} . Suppose $(x, y), (x, y') \in \mathcal{M}U$. Then

$$y'|W = x|W = y|W \quad \text{for all } W \in \mathcal{K}.$$

In particular, $y'|W = y|W$ for all $W \in \mathcal{K}$. Since \mathcal{G} is a sheaf and

$$U = \bigvee \{W \in \mathcal{K} \mid W \leq U\},$$

we get $y = y'$. Similarly we get

$$(x, y), (x', y) \in \mathcal{M}U \Rightarrow x = x'.$$

Moreover, let $y \in \mathcal{G}U$. $y|W \in \mathcal{G}W$, as W ranges over all elements of \mathcal{K} less than or equal to U , form a compatible family, and since $\mathcal{G}W = \mathcal{F}W$, it has a patch $x \in \mathcal{F}U$. Then, $(x, y) \in \mathcal{M}U$. It can also be easily verified that \mathcal{M} is a subsheaf of $\mathcal{F} \times \mathcal{G}$. thus \mathcal{M} is a graph of an isomorphism $\mathcal{F} \rightarrow \mathcal{G}$. (also see [18] theorem 3.4.10). \square

We are now starting with the construction. For each $J \in RidA$ let

$$S_J = \{s \in A \mid [s] \supseteq J\}.$$

Then, $J \mapsto A[S_J^{-1}]$ is a presheaf \mathcal{B} on $RidA$, the restriction maps $\nu_{IJ} : A[S_J^{-1}] \rightarrow A[S_I^{-1}]$ for $I \subseteq J$ resulting from the fact that $S_J \subseteq S_I$ whenever $I \subseteq J$, by the

universality property of the obvious homomorphisms $\nu_J : A \mapsto A[S_J^{-1}]$. Indeed we have

$$\begin{array}{ccc} A & \xrightarrow{\nu_J} & A[S_J^{-1}] \\ & \searrow \nu_I & \downarrow \nu_{IJ} \\ & & A[S_I^{-1}] \end{array}$$

where $\nu_{IJ}(\frac{a}{s}) = \nu_I(a)\nu_I(s)^{-1}$.

For each $c \in A$, $\mathcal{B}[c] = A[c^{-1}]$:

$$\begin{array}{ccc} A & \xrightarrow{\nu_c} & A[c^{-1}] \\ & \searrow \nu_{[c]} & \uparrow \varphi \downarrow \psi \\ & & A[S_{[c]}^{-1}] \end{array}$$

$[s] \supseteq [c] \iff c^n = sa$, some n, a . In such case $\alpha(c)$ invertible implies $\alpha(s)$ invertible for any ring homomorphism α on A . Thus, we get the maps φ and ψ as indicated in the diagram above since $\nu_c(s)$ is invertible for any $s \in S_{[c]}$ and $\nu_{[c]}(c)$ is invertible respectively. It can then be easily verified that φ and ψ are inverse to each other.

At each $J = [c]$, \mathcal{B} is separated and patching: Since $[c]$ is compact, it will suffice to verify this for finite covers. Furthermore, we can reduce this to covers of the form $[c] = [c_0] \vee \dots \vee [c_n]$, and by an induction argument, using that $[c] = [c_0] \vee [r_1 c_1 + \dots + r_n c_n]$ with suitable $r_i \in A$, it will be enough to consider the case $[c] = [b] \vee [d]$. But we have already seen the pullback lemma (in chapter 2) that the diagram

$$\begin{array}{ccc} A[c^{-1}] & \xrightarrow{\nu_{dc}} & A[d^{-1}] \\ \nu_{bc} \downarrow & & \downarrow \nu_{bdd} \\ A[b^{-1}] & \xrightarrow{\nu_{bdb}} & A[(bd)^{-1}] \end{array}$$

is a pullback whenever $[c] = [b] \vee [d]$. We get the required result since $A[c^{-1}] \cong A[S_{[c]}^{-1}]$.

For $J \in \text{Rid}A$, let

$$\mathcal{H}J = \{x \in \mathcal{B}J \mid J = \bigvee J_i \text{ with } x|J_i = 0\}.$$

Obviously $0 \in \mathcal{H}J$. Let $x \in \mathcal{H}J$ and $y \in \mathcal{B}J$. Then $J = \bigvee J_i$ with $x|J_i = 0$. Then $xy|J_i = (x|J_i)(y|J_i) = 0$. Furthermore, if $J = \bigvee J_i = \bigvee H_k$ with $x|J_i = 0 = y|H_k$, then $J = \bigvee_{i,k} J_i \wedge H_k$ and $(x+y)|J_i \wedge H_k = 0$. Thus $\mathcal{H}\mathcal{J}$ is an ideal of $\mathcal{B}\mathcal{J}$. It is also clear that $J \mapsto \mathcal{H}J$ is a subpresheaf of \mathcal{B} and that $\mathcal{B}_S J = \mathcal{B}J/\mathcal{H}J$ is the separated presheaf reflection of \mathcal{B} .

The presheaf map $\mu : \mathcal{B} \rightarrow \hat{A}[\hat{P}^{-1}]$: Let $a \in S_J$. Then $\bigvee \{\tilde{a}(b) \wedge [b] \mid b \in A\} = J \wedge [a] = J$. Thus $a \in \hat{P}J$. Thus we have a map

$$A \times S_J \rightarrow \hat{A}J \times \hat{P}J.$$

Let $\frac{a}{s} = \frac{a'}{s'}$ in $A[S_J^{-1}]$. Then $\exists t \in S_J$ such that $tas' = ta's$. Then $a \mapsto \tilde{a}$ being a ring homomorphism implies $\tilde{t}\tilde{a}s' = \tilde{t}\tilde{a}'\tilde{s}$. Then

$$\tilde{t} \in \hat{P}J \Rightarrow \frac{\tilde{a}}{\tilde{s}} = \frac{\tilde{a}'}{\tilde{s}'}$$

Thus we have got maps $A[S_J^{-1}] \xrightarrow{\mu} \hat{A}[\hat{P}^{-1}]J$,

$$\mu_J(\nu_J(a)\nu_J(s)^{-1}) = \left[\frac{\tilde{a}}{\tilde{s}} \right]_J,$$

for each $J \in \text{Rid}A$, where $\left[\frac{\tilde{a}}{\tilde{s}} \right]_J$ stands for the element of $\hat{A}[\hat{P}^{-1}]J$ determined by $(\tilde{a}, \tilde{s}) \in \hat{A}J \times \hat{P}J$.

(Explicitly, the map $A \rightarrow \hat{A}J$ takes S_J to $\hat{P}J$, leading to a map $A \times S_J \rightarrow \hat{A}J \times \hat{P}J$ which then induces the homomorphism $A[S_J^{-1}] \xrightarrow{\mu} \hat{A}[\hat{P}^{-1}]J$. Further, it is easily checked that this defines a presheaf map $\mathcal{B} \rightarrow \hat{A}[\hat{P}^{-1}]$).

The image of μ generates $\hat{A}[\hat{P}^{-1}]$: Let $\alpha \in \hat{A}J$ and $\beta \in \hat{P}J$. Then

$$J = \hat{\sigma}_J(\beta) = \bigvee \{\beta(s) \wedge [s] \mid s \in A\}.$$

Thus

$$\begin{aligned} J &= \bigvee \{\alpha(a) \mid a \in A\} \wedge \bigvee \{\beta(s) \wedge [s] \mid s \in A\} \\ &= \bigvee \{\alpha(a) \wedge \beta(s) \wedge [s] \mid a, s \in A\} \\ &\leq \bigvee \{\alpha(a) \wedge \beta(s) \mid a, s \in A\} \leq J. \end{aligned}$$

i.e.

$$J = \bigvee \{\alpha(a) \wedge \beta(s) \mid a, s \in A\}.$$

Let $I = \alpha(a) \wedge \beta(s) \leq J$. $\alpha|I$ is given by

$$\alpha : A \longrightarrow \downarrow J \longrightarrow \downarrow (\alpha(a) \wedge \beta(s)) \quad .$$

$$t \longmapsto \alpha(t) \longmapsto \alpha(t) \wedge \alpha(a) \wedge \beta(s)$$

Furthermore,

$$\alpha(t) \wedge \alpha(a) \wedge \beta(s) = \begin{cases} 0 & : a \neq t \\ \alpha(a) \wedge \beta(s) = I & : a = t \end{cases}$$

Thus, $\alpha|I = \tilde{a}$, and similarly we get $\beta|I = \tilde{s}$. Also: $[s] \supseteq I = \alpha(a) \wedge \beta(s)$ because $\beta(s) = \beta(s) \wedge J = \bigvee \{\beta(s) \wedge \beta(t) \wedge [t] \mid t \in A\} = \beta(s) \wedge [s] \leq [s]$, from which we get

$$\begin{aligned} \alpha(s) &= \alpha(s) \wedge J = \bigvee \{\alpha(s) \wedge \alpha(t) \wedge [t] \mid t \in A\} \\ &= \alpha(s) \wedge [s] \leq [s]. \end{aligned}$$

It follows that $s \in S_I$ and hence $\nu_I(a)\nu_I(s)^{-1} \in A[S_I^{-1}]$.

Also:

$$\begin{aligned} \left[\frac{\alpha}{\beta} \right]_J | I &= \left[\frac{\alpha|I}{\beta|I} \right]_I = \left[\frac{\tilde{a}}{\tilde{s}} \right]_I \\ &= \mu_I \left(\nu_I(a)\nu_I(s)^{-1} \right). \end{aligned}$$

Since the I cover J , this shows $Im(\mu)$ generates $\hat{A}[\hat{P}^{-1}]$.

$ker(\mu_J) = \mathcal{H}J$: Let $x \in ker(\mu_J)$. $x = \nu_J(a)\nu_J(s)^{-1}$, for some $a \in A$, $s \in S_J^{-1}$. Then

$$0 = \mu_J(x) = \mu_J \left(\nu_J(a)\nu_J(s)^{-1} \right) = \left[\frac{\tilde{a}}{\tilde{s}} \right]_J.$$

Then $J = \bigvee J_i$ and $t_i \cdot (\tilde{a}|J_i) = 0$ for some $t_i \in \hat{P}J_i$. Here $J_i = \bigvee \{t_i(c) \wedge [c] \mid c \in A\}$ and for each $I = t_i(c) \wedge [c] \neq [0]$, $\tilde{a}|I = 0$ and hence $ca = 0$. Further, $c \in S_I$ because $[c] \supseteq I$ trivially, hence $\nu_I(a)\nu_I(s)^{-1} = 0$, and since the I cover J , this shows $x \in \mathcal{H}J$. Conversely, $x = \nu_J(a)\nu_J(s)^{-1} \in \mathcal{H}J$ means $\nu_{J_i}(a) = 0$ for some cover $J = \bigvee J_i$ and then $t_i a = 0$ for some $t_i \in S_{J_i}$, hence $\tilde{t}_i \tilde{a} = 0$ and since $\tilde{t}_i \in \hat{P}J_i$ we have $[\frac{\tilde{a}}{\tilde{s}}] = 0$, saying $\mu_J(x) = 0$. \square .

It then follow from lemma 1 that

$$\tilde{\mu} : \tilde{B} \rightarrow \hat{A}[\hat{P}^{-1}]$$

is an isomorphism (\sim indicating the sheaf reflection); and from lemma 2 that

$$\tilde{B}[c] \cong B[c] \cong A[c^{-1}].$$

Thus

$$\hat{A}[\hat{P}^{-1}][c] \cong A[c^{-1}]$$

for any $c \in A$. In particular,

$$\hat{A}[\hat{P}^{-1}][1] \cong A[1^{-1}] \cong A.$$

As an added observation we can see that, using lemma 2.3 above we get

$$\mathcal{A} \cong \hat{A}[\hat{P}^{-1}],$$

where \mathcal{A} is the sheaf described in chapter 3.

Concluding Remarks:

(R1) In our work we simply observed that for a ring A , the map $A \rightarrow \text{Rid}A$, $a \mapsto [a]$ is a universal support. Here we would like to give a sketch of a 'natural' construction for this support. We will not recall basic definitions here and the reader is referred to [4, 11, 14, 20].

Construction: Let A be a ring and consider the following propositional theory.

Propositions: $a \in P$, one for each $a \in A$.

Axioms:

- (P0) $0 \in P \Rightarrow \perp$
- (P1) $\top \Rightarrow 1 \in P$
- (P2) $a \in P \wedge b \in P \iff ab \in P$
- (P3) $a + b \in P \Rightarrow a \in P \vee b \in P$.

One can then make A into a 'site' and get a flat map $A \rightarrow 'C - ideal(A)'$ as follows:

1. Define a preorder \leq on A , by

$$a \leq b \text{ iff } a^n = bc \text{ for some } n, c.$$

Note that in this preorder $ab \leq a$, in particular, $a^n = a$.

2. Define a covering system on (A, \leq) by

- (i) $\emptyset \in C(x)$ whenever $x^n = 0$ for some n ,
- (ii) $\{xa, xb\} \in C(x)$ whenever $x^n = a + b$ for some n .

Then it can be easily verified that the '*C-ideals*' for this theory are exactly the radical ideals of A . Thus if we let $SpecA$ = the locale (of primes of A) given by the above theory, then $\Omega SpecA = C - ideal(A) = RidA$. Hence a universal support $A \rightarrow RidA$, $a \mapsto [a]$.

Put differently, for basic propositions listed above, we can consider the set of all propositions generated from these by binary conjunction(\wedge) and arbitrary disjunction(\vee), with two distinguished propositions added: \top and \perp . The fundamental notion is the relation of entailment \vdash between propositions, subject to the following rules, where p, q, \dots stands for arbitrary propositions and S for any set of propositions.

1. Absolute: $p \vdash p$; if $p \vdash q$ and $q \vdash r$ then $p \vdash r$.
2. Conjunction:
 - if $p \vdash q$ and $p \vdash r$ then $p \vdash q \wedge r$.
 - $p \vdash \top$
 - $p \wedge r \vdash p, \quad p \wedge r \vdash r$
3. Disjunction:
 - $\vee S \vdash q$ whenever $p \vdash q$ for all $p \in S$.
 - $p \vdash \vee S$ for all $p \in S$.
 - $\top \vdash \vee \emptyset$.
4. Propositional law: $p \wedge \vee S \vdash \vee \{p \wedge q \mid q \in S\}$.

Axioms are presented as sets of entailments : $p \vdash q$, where p and q are specified propositions. Then the corresponding *Lindenbaum algebra* \mathcal{L} is given by taking the propositions of the theory modulo the relation $\vdash \dashv$ of mutual entailment, with partial order induced by \vdash . \mathcal{L} is then a frame, by the rules for entailment, and we have the map

$$\begin{aligned} \sigma : A &\longrightarrow \mathcal{L} \\ a &\mapsto [a \in P], \end{aligned}$$

where $[a \in P]$ is the element of the Lindenbaum algebra corresponding to ' $a \in P$ '. σ gives the universal model of the theory, and it can be shown that it is the universal support.

- (R2) Another approach to the generic sheaf representation of a ring is via the 'localization'/spectrum problem. intuitively, this says: Since the theory of rings is not equational, there is no universal local ring associated with

a given ring in a topos, particularly in Sets. However, with the ‘change of toposes’, Tierney in [19], Coste in [6] and Wraith in [23] showed that the inclusion of the category of local ringed toposes in the category of ringed toposes has a left adjoint, sending a ringed topos to what we call its spectrum. (By a (local)ringed topos we mean a topos with a particular (local)ring in it).

(R3) For the reader who want to study particular types of rings in terms of their pointfree spectra the following will be useful. Banaschewski in his papers [2, 3] has analysed the role played by $RidA$ in characterizing a given ring A . Among his findings are

- a ring is an exchange ring iff it is Gelfand with zerodimensional $JRidA$;
- a ring is Gelfand iff it is a ring of global elements of what we call a well-supported local ring in the topos of sheaves on a compact regular locale; and
- a ring is an exchange ring if it is the ring of global elements of a local ring in the topos of sheaves on a compact zerodimensional locale.

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